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A RAPID METHOD OF ESTIMATING THE COLLISION FREQUENCIES BETWEEN THE EARTH AND THE EARTH-CROSSING BODIES

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ABSTRACT

A very fast method of calculating the collision frequency between two low-eccentricity orbiting bodies has been developed for evaluating the evolution of Earth orbiting objects, such as space debris. The result is very accurate and the required computer time is negligible. This method is now applied without modification to calculate the collision frequencies for moderately and highly eccentric orbits. In collisions between the Earth and Earth-crossing bodies deviations in results are noticed when compared with the results published by Opik, Shoemaker et al., and Steel and Baggaley. The cause for some large deviations comes mainly from different methods used by different research groups in calculating the collision probabilities between ojects in similar inclinations. The errors in the results because of the approximation applied to the moderately and highly eccentric orbits are rather small. Furthermore, considering the tremendous computer time saved and the similar order of magnitude results, the fast method will be of great value in calculating the evolution of millions of objects circulating around a massive central attractor.

INTRODUCTION

The calculation of the collision probability between two objects orbiting around a massive central attractor, initiated by Opik /1/, has been extensively elaborated /2,3,4/. In general, the computation time for the collision integral used in all methods is enormous. In this report, a new calculational procedure is presented. This is based on expansion of the relative collision velocity of the orbiting objects in terms of orbital eccentricity under the assumption that the orbits have small eccentricities. The collision probability integral is then explicitly integrated for a few discrete orbital parameters so that tables are generated for subsequent interpolation (or extrapolation) to cover the whole range of orbital parameters. Although the calculation procedure is strictly valid only for low eccentricity orbits, it will be applied without modification to calculate the collision probabilities between orbiting objects of all eccentricities.

ANALYSIS

Following Kessler's approach /4/, the average number of collisions N per time Δt between two small orbiting objects is given by

$$N/\Delta t = \bar{\sigma} \iint S_1 \; S_2 \; V_r (2\pi R^2 \cos\beta \; d\beta \, dR), \qquad (1)$$

where $$S_j = 1 \Big/ \left\{ 2\pi^3 a_j R[(q_j' - R)(R - q_j)(\sin^2 i_j - \sin^2\beta)]^{1/2} \right\}, \qquad j = 1,\ 2. \qquad (2)$$

Here the subscript j denotes either the orbiting object 1 or 2. The spatial density, S, is related to the object's orbital elements a, q', q, and i (the semi-major axis, the apocentron, the pericentron and the inclination, respectively). Note that the average collision cross-section for the two colliding objects, $\bar{\sigma}$ (which includes the gravitational focussing effect), has beeen taken out of the integral in equation (1). V_r is the relative velocity between the two orbiting objects. R and β are, respectively, the radial distance and the latitude angle of the orbiting object with the massive central attractor at the center of a reference coordinate system. The volume integral covers all the common space accessible to both objects. Thus the space of integration is confined between $q'_{min} > R > q_{max}$, and $i_{min} \geq \beta \geq 0$. The subscripts max and min denote, respectively, the maximum and the minimum of the orbital elements of the two objects considered. It is noticed that equation (1) becomes improper when either $q_1' = q_2'$, or $q_1 = q_2$, or $i_1 = i_2$, or $i_1 = 180^\circ - i_2$ for non-vanishing $\bar{\sigma}$ and V_r. These singularities have been pointed out and been removed by using distributed orbital elements by Wetherill /2/ and Su /5/.

The collision probability integral in equation (1) can only be integrated analytically when the two orbits are circular and have identical inclination. For all other cases, numerical integration is needed. Because of the large common space covered by the two orbits, enormous computation time is required. The computation time grows as the square of the number of the orbiting objects involved. However, when the orbits have very small eccentricities, equation (1) can be approximated by expanding the relative velocity V in terms of the orbital eccentricity as a small parameter and equation (1) can be greatly simplified.

Thus for every orbit, assuming that the eccentricity is small, the orbital velocity V is first expanded in terms of eccentricity and the resulting relative velocity is then averaged over the entire orbits of the the two colliding objects. The relative velocity is then reduced to

$$V_r = [\, V_{c1}^2 + V_{c2}^2 - 2\, V_{c1} V_{c2} \cos(\alpha_1 - \alpha_2)\,]^{1/2} \qquad (3)$$

where $\cos\alpha_j = \cos i_j / \cos\beta$, $\qquad j = 1, 2.$

The velocity V_{cj} is the circular velocity of the object j based on its semi-major axis. It should be noted that the first order term in eccentricity disappears after the averaging process and all terms higher than second order are ignored completely in equation (3). Thus, we have to keep in mind that errors can apppear in the collision frequency in the case when $a_1 = a_2$, and $i_1 = i_2$ resulting in a vanishing V_r in equation (3). Substituting equation (3) into equation (1), the collision probability integral becomes

$$\frac{N}{\Delta t} = \frac{\sigma V_{c1} (1 + a_1/a_2)^{1/2}}{\pi^5 a_1 a_2}\, X \; Y , \qquad (4)$$

with

$$X = \frac{2}{[(q'_{max} - q_{max})(q'_{min} - q_{min})]^{1/2}} \; K(Q) , \qquad (5)$$

$$K(Q) \approx 1.57078 - 0.25701\, Q + 0.07252\, Q^2 + (0.65021 - 0.17908\, Q + 0.02887\, Q^2)\, \ln \frac{1}{1-Q} , \qquad (6)$$

$$Q = \frac{(q'_{max} - q_{min})(q'_{min} - q_{max})}{(q'_{max} - q_{max})(q'_{min} - q_{min})} , \qquad (7)$$

$$Y = \frac{1}{2} (Y^+ + Y^-) , \qquad (8)$$

$$Y^{\pm} = \int_0^{i_{min}} \frac{[\cos^2\beta - A \cos i_1 \cos i_2 \pm A (\sin^2 i_1 - \sin^2\beta)^{1/2} (\sin^2 i_2 - \sin^2\beta)^{1/2}]^{1/2}}{(\sin^2 i_1 - \sin^2\beta)^{1/2} (\sin^2 i_2 - \sin^2\beta)^{1/2}}\, d\beta , \qquad (9)$$

$$A = \frac{2 (a_1 a_2)^{1/2}}{a_1 + a_2} . \qquad (10)$$

The function K which appears in equation (5) is the complete elliptic integral of the first kind. For quick reference, the polynomial approximation of the function K from the Handbook of Mathematical Functions /6/ is repented in equation (6).

The four singularities existed in equation (1) are still presented in equation (4) (two in function X and two in function Y). To obtain finite collision probabilities between two obital objects, we can first numerically integrate integral Y, generating tables for subsequent interpolation of the function Y for other a's and i's. This is especially useful

Table 1(a)-(c). Angle Integral of the Collision Probability Integral (Value of Y in Eq.9)

(a) $2(a_1 a_2)\ /(a_1+a_2)=0.99$

	0		20		40		60		80		100		120		140		160		180
180	5268.71*																		
	12.69	26.86*																	
160	6.37	6.85	15.06*																
	4.27	4.41	4.98	10.85*															
140	3.23	3.30	3.51	4.03	8.72*														
	2.62	2.65	2.76	2.99	3.48	7.48*													
120	2.21	2.24	2.30	2.43	2.67	3.13	6.72*												
	1.93	1.95	1.99	2.08	2.22	2.46	2.91	6.26*											
100	1.72	1.73	1.77	1.83	1.93	2.08	2.32	2.78	6.01*										
	1.57	1.58	1.61	1.66	1.73	1.84	2.00	2.26	2.73	6.01*									
80	1.45	1.46	1.48	1.52	1.58	1.67	1.80	1.98	2.79+	2.73	6.01*								
	1.36	1.36	1.38	1.42	1.47	1.55	1.66	2.36+	1.98	2.26	2.78	6.26*							
60	1.29	1.29	1.31	1.34	1.39	1.47	2.16+	1.66	1.80	2.00	2.32	2.91	6.72*						
	1.23	1.24	1.26	1.29	1.35	2.09+	1.47	1.55	1.67	1.84	2.08	2.46	3.13	7.48*					
40	1.20	1.21	1.23	1.27	2.12+	1.35	1.39	1.47	1.58	1.73	1.93	2.22	2.67	3.48	8.72*				
	1.18	1.19	1.23	2.30+	1.27	1.29	1.34	1.42	1.52	1.66	1.83	2.08	2.43	2.99	4.03	10.85*			
20	1.21	1.24	2.77+	1.23	1.23	1.26	1.31	1.38	1.48	1.61	1.77	1.99	2.30	2.76	3.51	4.98	15.06*		
	1.42	4.31+	1.24	1.19	1.21	1.24	1.29	1.36	1.46	1.58	1.73	1.95	2.24	2.65	3.30	4.41	6.85	26.86*	
0	372.84+	1.42	1.21	1.18	1.20	1.23	1.29	1.36	1.45	1.57	1.72	1.93	2.21	2.62	3.23	4.27	6.37	12.69	5268.71*

i_1 (Inclination of Object 1, degree)

(b) $2(a_1 a_2)\ /(a_1+a_2)=0.97$

	0		20		40		60		80		100		120		140		160		180
180	5242.16*																		
	12.63	26.72*																	
160	6.34	6.82	14.99*																
	4.25	4.39	4.95	10.79*															
140	3.22	3.28	3.49	4.02	8.68*														
	2.61	2.64	2.75	2.98	3.46	7.44*													
120	2.21	2.23	2.30	2.43	2.66	3.12	6.69*												
	1.93	1.94	1.99	2.08	2.22	2.45	2.90	6.24*											
100	1.72	1.73	1.77	1.83	1.93	2.08	2.33	2.78	6.01*										
	1.57	1.58	1.61	1.66	1.73	1.85	2.01	2.28	2.77	6.43*									
80	1.45	1.46	1.48	1.53	1.59	1.68	1.82	2.03	3.23+	2.77	6.01*								
	1.36	1.37	1.39	1.43	1.49	1.57	1.71	2.83+	2.03	2.28	2.78	6.24*							
60	1.30	1.31	1.33	1.36	1.42	1.52	2.68+	1.71	1.82	2.01	2.33	2.90	6.69*						
	1.26	1.26	1.29	1.33	1.41	2.67+	1.52	1.57	1.68	1.85	2.08	2.45	3.12	7.44*					
40	1.24	1.25	1.28	1.35	2.82+	1.41	1.42	1.49	1.59	1.73	1.93	2.22	2.66	3.46	8.68*				
	1.25	1.27	1.35	3.20+	1.35	1.33	1.36	1.43	1.53	1.66	1.83	2.08	2.43	2.98	4.02	10.79*			
20	1.36	1.43	4.08+	1.35	1.28	1.29	1.33	1.39	1.48	1.61	1.77	1.99	2.30	2.75	3.49	4.95	14.99*		
	1.91	6.89+	1.43	1.27	1.25	1.26	1.31	1.37	1.46	1.58	1.73	1.94	2.23	2.64	3.28	4.39	6.82	26.72*	
0	645.77+	1.91	1.36	1.25	1.24	1.26	1.30	1.36	1.45	1.57	1.72	1.93	2.21	2.61	3.22	4.25	6.34	12.63	5242.16*

i_1 (Inclination of Object 1, degree)

(c) $2(a_1 a_2)\ /(a_1+a_2)=0.95$

	0		20		40		60		80		100		120		140		160		180
180	5215.48*																		
	12.56	26.59*																	
160	6.31	6.79	14.91*																
	4.23	4.37	4.93	10.74*															
140	3.21	3.27	3.48	4.00	8.63*														
	2.60	2.63	2.74	2.97	3.45	7.41*													
120	2.20	2.22	2.29	2.42	2.65	3.11	6.66*												
	1.92	1.94	1.98	2.07	2.21	2.45	2.90	6.22*											
100	1.72	1.73	1.77	1.83	1.93	2.08	2.33	2.79	6.00*										
	1.57	1.58	1.61	1.66	1.74	1.85	2.02	2.29	2.80	6.73*									
80	1.45	1.46	1.49	1.53	1.60	1.69	1.83	2.06	3.55+	2.80	6.00*								
	1.37	1.38	1.40	1.44	1.50	1.60	1.75	3.16+	2.06	2.29	2.79	6.22*							
60	1.31	1.32	1.34	1.38	1.45	1.57	3.04+	1.75	1.83	2.02	2.33	2.90	6.66*						
	1.28	1.29	1.31	1.36	1.47	3.09+	1.57	1.60	1.69	1.85	2.08	2.45	3.11	7.41*					
40	1.27	1.28	1.33	1.42	3.31+	1.47	1.45	1.50	1.60	1.74	1.93	2.21	2.65	3.45	8.63*				
	1.32	1.34	1.44	3.84+	1.42	1.36	1.38	1.44	1.53	1.66	1.83	2.07	2.42	2.97	4.00	10.74*			
20	1.50	1.59	5.02+	1.44	1.33	1.31	1.34	1.40	1.49	1.61	1.77	1.98	2.29	2.74	3.48	4.93	14.91*		
	2.29	8.70+	1.59	1.34	1.28	1.29	1.32	1.38	1.46	1.58	1.73	1.94	2.22	2.63	3.27	4.37	6.79	26.59*	
0	833.68*	2.29	1.50	1.32	1.27	1.28	1.31	1.37	1.45	1.57	1.72	1.92	2.20	2.60	3.21	4.23	6.31	12.56	5215.48*

i_1 (Inclination of Object 1, degree)

* Due to singularities at $i_1+i_2=180°$, these entris assume $i_1+i_2=179.9°$.
\+ Due to signularities at $i_1=i_2$, these entries assume $i_1=i_2-1\times10^{-4}$

when the mutual collision probability between thousands of orbiting objects is to be calculated. As a simple illustration, values of Y for different combinations of i_1 and i_2 are tabulated in Table 1(a) through 1(c) for the special cases of A = 0.99, 0.97, and 0.95, respectively. Because the value of Y becomes infinite at $i_1 = 180° - i_2$ for all A's and at $i_1 = i_2$ for $A < 1$, the entry of Y at $i_1 + i_2 = 180°$ is replaced by the value of Y at $i_1 + i_2 = 179.9°$ and the entry at $i_1 = i_2$ by the value at $i_1 = i_2 - 0.0001$ for interpolation purposes.

APPLICATION TO THE EARTH-CROSSING OBJECTS

The collision probability between the Earth and an Earth-crossing body can now be readily calculated from equation (4) with aids of equations (5) and (6) and Tables 1(a) through 1(c).

Table 2. Collision Probabilitics of Aten and Apollo Asteroids with the Earth

NAME	ELEMENTS a (A U)	e	i (deg)	COLLISION PROBALILITY PER 10^9 YEARS (Opik)	(SWHW)	(SB)	(this paper)
Aten Asteroids							
Hathor	0.8439	0.4498	5.86	14	14	14.71	20.49*
Ra-Shalom	0.8321	0.4364	15.76	6.3	6.7	6.11	3.19
Aten	0.9665	0.1826	18.94	6.9	6.4	8.01	5.05
1954XA	0.7772	0.3454	3.93			59.85	210.20*
Apollo Asteroids							
1983TB	1.2715	0.8903	22.04			1.38	0.43
Icarus	1.0779	0.8268	22.91	1.6	2.0	1.77	0.65
Hephaistos	2.1637	0.8351	11.89			1.05	0.76
1974MA	1.7752	0.7620	37.79			0.56	0.31
Adonis	1.8749	0.7638	1.36			11.53	37.68*
1972TA	2.3030	0.7710	12.12			0.99	1.04
Daedalus	1.4609	0.6148	22.16	1.0	1.6	1.27	0.58
Cerberus	1.0801	0.4669	16.09	2.5	3.1	3.14	1.31
Hermes (1937 UB)	1.6393	0.6236	6.22	2.2	3.7	3.50	21.20*
Midas	1.7759	0.6499	39.84	3.8	0.7	0.65	0.41
1974XC	2.1734	0.7118	2.52			5.37	63.30*
1981VA	2.4600	0.7439	22.02			0.56	0.42
Apollo	1.4712	0.5600	6.35			4.30	24.10*
1979XB	2.2624	0.7133	24.87			0.60	0.39
Bacchus	1.0776	0.3495	9.42			6.69	2.43
1983LC	2.6316	0.7092	1.52			7.54	158.81*
Toro	1.3672	0.4359	9.37			4.22	8.91
Aristaeus	1.5996	0.5037	23.04			1.44	0.80
6743P-L	1.6805	0.5237	7.90			3.46	23.60*
1983VA	2.6143	0.6925	16.25			0.82	1.56
1982HR	1.2100	0.3227	2.69			23.82	33.03*
Orthos	2.4042	0.6586	24.39			0.71	0.60
1983TF2	1.3428	0.3871	7.84			6.09	30.33*
Geographos	1.2446	0.3355	13.32			4.49	2.15
Sisyphus	1.8930	0.5394	41.15			0.96	0.67
1973NA	2.4272	0.6381	68.00			0.61	0.45
1978CA	1.1248	0.2148	26.12			4.45	3.13
Antinous	2.2602	0.6065	18.42			1.21	1.29
Tantalus	1.2900	0.2984	64.02	2.5	1.5	2.50	2.13
1982BB	1.4070	0.3548	20.94			3.09	1.85
6344P-L	2.6186	0.6411	4.65			4.72	244.18*
1982DB	1.4893	0.3602	1.42			63.26	273.51*
1979VA	2.6354	0.6273	2.78			16.71	965.36*
1950DA	1.6834	0.5020	12.15			2.53	1.87

* Large differencies in these Collision probabilities between SB and the current method come from the existence of the singularity in the collision integval in equation (1) considered in the current method and ignored by SB.

Lagrangian interpolation (and extrapolation) is used to obtain the Y function from Tables 1(a) through 1(c). Although the value A may lie far outside the range of the values of A in the tables, the extrapolation is still applied. The reason for such an approach is based on the observation that the value of Y does not change too drastically when eqution (6) behaves properly. However, for orbital elements that fall near the singularities in the tables, large collision probabilities will result and the result should be treated with care.

Table 2 lists the results of the collision frequencies between the Earth and the Earth-crossing bodies obtained by Opik, Shoemaker et al., Steel and Baggaley /7/ and the current method. At a first glance, one may think that extremely large differences exist between the results obtained by the current method and other published results. However, let us first devide the asteroids into two groups, one with asteroids having high orbital inclinations ($i > 10^\circ$) and the other, low inclinations ($i \leqslant 10^\circ$) and then make comparison. We would find that the current method yields very favorable results for the asteroids with high inclinations when compared with the published results. The small deviations in the current results may be contributed to the expansion approximation of the collision velocity in terms of orbital eccentricity used in the current method.

On the other hand, extremely large differences in the collision probabilities for the asteroids with low inclinations between the current result and other published results, notably that of Steel and Baggaley, does not come from errors in the eccentricity expansion approximation used in the current method. Rather, the differences come from the existing of the singularity (at the Earth orbital inclination of $i = 0.016^\circ$) in the collision probability in equation (1) considered in the current method and ingored by Steel and Baggaley using volume averaging method. Under such circumstances, comparison of the results is meaningless. Nevertheless, we would like to emphasize the existence of the singularities in the collision probability integral when the discrete orbital elements are used. The singularities can be removed by using distributed orbital elements in the probability integral /5/ and can not removed by using discrete finite volume summation method.

As was stated in the beginning, the current method is intended to yield a quick estimate of the collision probabilities among many colliding objects. The computer time needed to obtain the collision probabilities for the 39 Aten asteroids and the Earth in Table 2 is just a couple of seconds in PC-386 with 30387.

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