The purpose of this exercise is to construct the “9-point” circle related to a triangle. While we begin with a scalene, acute triangle, we will later look at the 9-point circle and the equivalent Euler line for the acute isosceles, equilateral, right (both scalene and isosceles), and scalene obtuse triangles. In doing so, we will examine the properties of the altitudes, orthocenter, perpendicular bisectors, circumcenter, circumcircle, medians, centroid, and Euler line as they relate to these different types of triangles. We will demonstrate that the orthocenter, circumcenter, centroid, and center of the 9-point circle are all collinear.

Construction: Step 1 – Create the Orthocenter

First we construct ΔABC. Then we construct the altitudes to each side of ΔABC. These altitudes are concurrent at the orthocenter H and intersect the sides at points D, E, and F. (See below)
Construction: Step 2 – Construct the circumcenter and circumcircle.

Next we construct the three \( \perp \) bisectors of the vertex angles of \( \triangle ABC \). We do this by first constructing the midpoints of the three sides of \( \triangle ABC \) and labeling these points \( A' \), \( B' \) and \( C' \). We then construct \( \perp \) lines through these midpoints. These three lines are concurrent at the point labeled \( O \). This is the circumcenter, or the center of the circumcircle that intersects points \( A \), \( B \), and \( C \). This can be seen in the next construction.
Construction: Step 3 - Create the Euler Line

Next, let's hide the \( \perp \) bisectors of \( \triangle ABC \) and bring our 3 altitudes back. We now can clearly see the points H (the orthocenter) and O (the circumcenter). The Euler line connects these two points (by definition). We will construct the Euler line HO with its midpoint as well.

Since the endpoints of the Euler line, the orthocenter and circumcenter, are “within” the triangle, the Euler line is also within the triangle.
Construction: Step 4 – Construct the centroid

Next, we will construct the centroid (G) of our triangle. Since the centroid is the intersection of the medians of the triangle, we just connect the vertices with the respective midpoints, thus creating segments AA’, BB’, and CC’.

While it appears that H, G, N, and O are collinear, we will show this a little later. By definition, point N (the midpoint of HO) is on this line, so this proof will just require that we locate G on the line HO.
**Construction**: Step 5 – Construct the midpoints from orthocenter to vertices

Now let us create the last three points in our 9-point circle. These last three points are the midpoints of the distance from the orthocenter (O) to each vertex A, B, and C. These three points, labeled K, L, and M are constructed below.

Having constructed all 9 of the points that “should” be cocircular on the 9-point circle, let us now finally construct that circle.
Construction: Step 6 – Construct the 9-point circle, demonstrate 9 points are cocircular

The 9-point circle has its center at the midpoint of the Euler line HO (point N) and it radius is equal to the distance of that point to each of the 9 points D, A’, L, C’, F, K, B’, E, and M. This circle appears in the following construction.

As one can see, it appears that the 9-point circle of our acute, scalene triangle intersects all 9 constructed points. Measuring the distance from point N to each of the 9 points proves this. As we can see, this is 2.65 cm. for all 9 points. The 9-point circle has a radius less than that of the circumcircle since all 9 points are either on, or within the triangle.
Construction: Step 7 – Demonstrate that orthocenter, circumcenter, and centroid are collinear

Now, let's prove that the centroid (G) is collinear with the other three points of the Euler line, the orthocenter (H), the circumcenter (O) and the midpoint (N). To do this, we will need to add a Cartesian coordinate system to our construction. We will allow point A to reside at the origin. We will then use the coordinates of points H and O to measure the slope of segment HO. If the slope of HG has the same slope, then G is collinear with H, N, and O.

As one can see, the slope of HO is -1.64 as is the slope of HG. This means that these points are all collinear (again, N is collinear by construction since it is the midpoint of the Euler line).

Summary: We have completed our construction with an acute, scalene triangle. This shows that the Euler line is “within” the circle (because the orthocenter is inside the triangle). The orthocenter, circumcenter, centroid, and center of the 9 point circle are all collinear. The 9 point circle is smaller than the circumcircle.
Constructing: Step 8: The 9-point circle using a scalene, obtuse triangle

Using the same Geometer’s sketchpad model, we now create the 9-point circle using a scalene, obtuse triangle. As one can see, the orthocenter (H) and circumcenter (O), are both outside triangle ΔABC. Thus the Euler line is much longer (relative to the triangle) than in our previous case. The centroid (G) will always be within the triangle since it is the concurrent point of the medians of the triangle. Note also that we still have 9 distinct points, but 5 of the 9 points are outside the triangle. Points D, and E (altitude intersections) intersect the extension of sides BC and AC (respectively) outside the triangle.
Constructing: Step 9: The 8-point circle using an acute isosceles triangle.

Using the same Geometer’s sketchpad model, we now create the 9-point circle using an acute isosceles triangle. The orthocenter (H) and the circumcenter (O) are is at the midpoint of the hypotenuse. The Euler line is collinear with the median from the vertex to the base of the isosceles triangle. The altitude and perpendicular bisectors to the base are the same, so the intersection of those two lines with the base of the triangle is concurrent. Thus our 9-point circle intersects 8 distinct points. The obtuse isosceles triangle also has 8 points in its 9-point circle.
Constructing: Step 10: The 6-point circle using an equilateral triangle

In the equilateral triangle below, we see that the Orthocenter, centroid, and circumcenter become concurrent, so that the Euler line has a length of 0. Further, the altitudes and medians are concurrent, so that the intersections with the sides (A', B', C', and D, E, F) are concurrent. The 9-point circle now contains only 6 points.
**Constructing: Step 11: The 5-point circle using a scalene right triangle**

Using the same Geometer’s sketchpad model, we now create the 9-point circle using a scalene right triangle. The orthocenter (H) is at the vertex of the right angle and the circumcenter (O) is at the midpoint of the hypotenuse. Thus the Euler line is now the median from the right angle to the hypotenuse. Since we have a right angle, the points K merges with the right angle vertex, and points L and M are concurrent with the midpoints of the two base sides of the triangle. Also two of the altitude intersections are also concurrent with the midpoints of the base sides. Only point D (altitude intersection from the right angle vertex to the hypotenuse) is a separate point. Thus our 9 point circle only intersects 5 distinct points, all of which are on the triangle.
Constructing: Step 12: The 4-point circle using an isosceles right triangle

An isosceles right triangle has the same characteristics as the scalene right triangle, with the exception that the altitude and median from the right angle vertex to the hypoteneuse become collinear. Thus, points A’ and D become concurrent. This reduces the 5-points in the scalene right triangle to just 4 points.

Summary: In the first 7 construction steps we created the GSP model for the 9-point circle, demonstrating that it met the requirements outlined in our project. In the succeeding steps, we saw how the shape and behavior of the various components (Euler line, circumcircle, circumcenter, centroid, medians, altitudes, and 9-point circle) changed. We could see that the 9-point circle can have nine, eight, six, five, or four distinct points, depending on whether the triangle is scalene, right, or equiangular, and scalene, isosceles, or equilateral.