Spherical Geometry – Geometry Final Part B – Problem 4

By: Douglas A. Ruby
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Grades: 11/12

Problem 4: Arc length and the area of a triangle in spherical geometry are quite different from that done in Euclidean geometry. Assume the radius of the Earth to be 3960 miles. Three positions on the surface of the Earth are defined by:

i) Latitude = 0°, longitude = 60°
ii) Latitude = 90°, longitude = 60°
iii) Latitude = 0°, longitude = 90°

a) Find the perimeter and area of the spherical triangle.
b) Find the area and perimeter of the triangle whose vertices are at the three points assuming Euclidean geometry.
c) How do the areas in Part a) and Part b) differ?
Note: Use software support as necessary

1. Diagramming the Problem

Using Geometer’s sketchpad, we will try to approximate a sphere by drawing a circle with a vertical diameter representing the Greenwich meridian. We will indicate the equator with an arc created using three points, two on the circle and one on the Greenwich Meridian.
Observations: The three given points, A (on the equator at 60° East), B (at the North Pole), and C (on the Equator at 90° East), are indicated. The viewpoint is from approximately 30° North, directly over the Greenwich Meridian. Spherical ΔABC is indicated in bold lines, while Euclidean ΔABC is shown in dotted lines. We also see the projection of segments AB and CB onto the Equatorial plane as OA and OC in dotted lines. Note that OA, OB, and OC are radii of the Earth with length r = 3960 miles. Finally, given the position of points A and C on the equator, ∠AOC has a measure of 30°. ∠AOC is also the projection of spherical ∠BOC onto the equatorial plane. Angles ∠BAC = ∠BCA = 90°.

a) Find the perimeter and area of the spherical triangle.

The perimeter of our spherical triangle can be calculated by using the basic formula for the circumference of a circle:

1. \( C = D\pi = 2\pi r \) (where \( D = \) diameter and \( r = \) radius)

In our case, \( r = 3960 \) miles. Thus, a great circle circumference is:

2. \( 2 \times 3960 \) miles \( \times \pi \approx 24,881.413816431162448624135595574 \) miles

Since both spherical segments BA and BC are from the North Pole to the Equator, \( \angle BOA \) and \( \angle BOC \) are both = 90°. Thus:

3. \( BA = BC = \frac{90°}{360°} \times 2 \times 3960 \times \pi \approx 6,220.3534541077906121560338988934 \) miles

Spherical segment AC, is a 30° arc on the equator. Thus, its length is:

4. \( AC = \frac{30°}{360°} \times 2 \times 3960 \times \pi \approx 2,073.4511513692635373853446329645 \) miles.

Adding the results of equations 3 and 4, we get the perimeter of spherical ΔABC which is:

5. \( BA + BC + AC = \text{perimeter of spherical } \triangle ABC \approx 14,514.158 \) miles

To find the area (K) of spherical ΔABC, we use the formula:

6. \( K = \frac{\pi r^2}{180} \left( A + B + C - 180° \right) \) (where A, B, and C are the measures of the vertex angles of the spherical triangle; which will always be greater than 180°)
Using \( A = C = 90^\circ, B = 30^\circ, \) and \( r = 3960 \) miles, for the \textit{Area of our spherical triangle, we get}:

\[
7. \quad K = \frac{\pi \times 3960^2}{180} (90 + 30 + 90 - 180) = \frac{\pi \times 3960^2}{60} \approx 8,210,866.559 \text{ sq. miles}
\]

b) \textbf{Find the area and perimeter of the Euclidean triangle.}

To calculate the dimensions of the Euclidean triangle, we can either use Euclidean Geometry with the Pythagorean Theorem or use Trigonometry. Since our comparison is Euclidean versus Spherical, we will use traditional triangulation without having to know the values of the sine, cosine, or tangents of the angles formed by the various triangles.

First, let’s look at the projections of \( \triangle AOB \) and \( \triangle COB \) across their respective longitudinal planes (60° East and 90° East respectively):

Triangle \( \triangle COB \) is a 45/45/90 right triangle. Since the legs (OC and OB) are radii of the Earth, the formula for the length of two sides of Euclidean triangle \( \triangle ABC \) is:

\[
8. \quad AB = BC = 3960\sqrt{2} \approx 5,600.2857069974563932546873478704 \text{ miles}
\]

Finding the length of segment AC will be a bit more difficult. To do this, we look at a projection of \( \triangle ABC \) onto the Equatorial plane:
Angle ∠COA is a $30^\circ$ angle, therefore, ΔDOA is a 30/60/90 triangle. Segment OA is 3960 miles. From this, we can calculate that:

9. $AD = \frac{3960}{2} = 1,980$ miles. $OD = 1980\sqrt{3} \approx 3,429.4605989863770411843437561816$ miles

10. $CD = (OC - OD) = 530.53940101362295881565624381837$ miles

11. $AC = \sqrt{AD^2 + CD^2} = \sqrt{1980^2 + 530.5394^2} \approx 2,049.8468372119644378032787939825$ miles

Thus, we now know that the perimeter of Euclidean ΔABC is:

12. $AB + BC + AC = (2 \times 5,600.2857$ miles $) + 2,049.8468$ miles $= 13,250.418$ miles
To find the area of Euclidean ΔABC, we need to look at a projection of ΔABC through longitude 75°, ½ of the way between 60° and 90°.

![Diagram of spherical geometry with points A, B, C, and E labeled.]

We can find the height of ΔABC (segment EB) by using the Pythagorean Theorem. Once we do that, we will find the area of ΔABC using the standard formula.

13. $EB = \sqrt{5600.2857^2 + \left(\frac{2049.8468}{2}\right)^2} \approx 5,693.3002743581840220998243073757$ miles

14. $Area_{ABC} = \frac{1}{2}bh = \frac{1}{2}AC \times EB \approx 5,835,196.78$ sq. miles

From this diagram, we can also calculate the measure of $\angle A$, $\angle B$, and $\angle C$ using trigonometric functions. Please see the summary for the results of this.
iii) **Summary**

Let’s start with a quick comparison of our Spherical and Euclidean triangles:

<table>
<thead>
<tr>
<th>Triangle</th>
<th>BA=BC</th>
<th>AC</th>
<th>Perimeter</th>
<th>Area</th>
<th>Sum of ∠’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spherical</td>
<td>6,220.35 miles</td>
<td>2,073.45 miles</td>
<td>14,514.16 miles</td>
<td>8,210,867 miles²</td>
<td>210°</td>
</tr>
<tr>
<td>Euclidean</td>
<td>5,600.29 miles</td>
<td>2,049.85 miles</td>
<td>13,250.42 miles</td>
<td>5,835,197 miles²</td>
<td>180°</td>
</tr>
</tbody>
</table>

We have shown, using Spherical geometry as well as traditional Euclidean (non-Trig) geometry that even given the same vertices in 3-space, the sum of the interior angles of the spherical triangle is greater than the Euclidean triangle. The Spherical triangle has a larger area and a larger perimeter. We have also shown how to solve the 3-space Euclidean triangle using 2-space projections.

I might also note, that in the case of our particular spherical triangle, the chosen points made solving the area problem much easier, since we did not have to calculate the φ, δ, and θ with respect to the origin of the sphere. On the other hand, calculating the vertex angles for the Euclidean triangle requires the use of the trigonometric arctangent function.

From our last diagram, used to calculate the area of Euclidean ΔABC, we can see that:

\[
∠A = ∠C = \tan^{-1}\left(\frac{5693.300}{\frac{1}{2} \times 2049.85}\right) \approx 79.795° \\
∠B = 2 \times \tan^{-1}\left(\frac{\frac{1}{2} \times 2049.85}{5693.300}\right) \approx 20.410°
\]

A summary of the differences in the vertex angles between our spherical and Euclidean triangle follows:

<table>
<thead>
<tr>
<th>Triangle</th>
<th>∠A</th>
<th>∠B</th>
<th>∠C</th>
<th>Sum of ∠’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spherical</td>
<td>90°</td>
<td>30°</td>
<td>90°</td>
<td>210°</td>
</tr>
<tr>
<td>Euclidean</td>
<td>79.795°</td>
<td>20.410°</td>
<td>79.795°</td>
<td>180°</td>
</tr>
</tbody>
</table>