DEVELOPING A ROUTINE TO POLAR ALIGN A TELESCOPE

Assumptions:

It is assumed that the reader is familiar with a monograph by Toshimi Taki, (http://www.asahi-net.or.jp/~zs3t-tk/), especially Section 4 wherein he details the use of Direction Cosines to describe a telescope and the various reference frames that can be used. Knowledge of matrix manipulation and basic trigonometric identities is also assumed.

Approach:

We want to derive equations to describe the change in the equatorial coordinates **(** $-H, \delta$ **)** a telescope will be pointing at if we move the scope's polar axis away from the North (or South) celestial pole by θ in azimuth and γ in elevation, but keep the telescope's Declination and Right Ascension axes fixed. With these equations, we should be able to accomplish the following:

1. Measure the degree of polar axis misalignment in azimuth and elevation.
2. Apply a correction from a known star position that can be used to align the polar axis. i.e. if we know how much the equatorial coordinates change when introducing a polar alignment error, we should be able to point a telescope at such an offset and move the polar axis until the object is in the eyepiece, thus aligning the polar axis.

Figure 1-1 shows the axes of a polar aligned telescope and the optical axis directed at an object with equatorial coordinates (**-H**, δ). On a unit sphere, these coordinates can be expressed as Direction Cosines, (L_e, M_e, N_e)

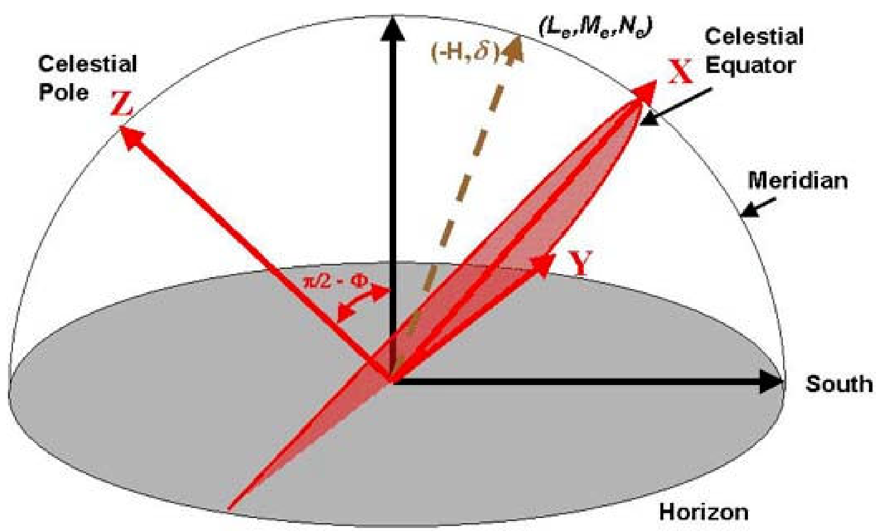

Figure 1-1

The Direction Cosines for an equatorial mount can be expressed as:

$$\begin{pmatrix} \boldsymbol{L_e} \\ \boldsymbol{M_e} \\ \boldsymbol{N_e} \end{pmatrix} = \begin{pmatrix} \cos\delta\cos(-H) \\ \cos\delta\sin(-H) \\ \sin\delta \end{pmatrix} \qquad \text{…Eq 1-1}$$

Where:

δ **=** Declination

$\boldsymbol{H}$ **=** Hour Angle = current sidereal time – Right Ascension

In this reference frame, the Z-axis is directed at the celestial pole
the Y-axis is directed East
the X-axis is directed South

For the misaligned mount above, the Direction Cosines become:

$$\begin{pmatrix} \boldsymbol{L_e'} \\ \boldsymbol{M_e'} \\ \boldsymbol{N_e'} \end{pmatrix} = \begin{pmatrix} \cos\delta'\cos(-H') \\ \cos\delta'\sin(-H') \\ \sin\delta' \end{pmatrix} \qquad \text{…Eq 1-2}$$

Where:

$\delta' = \delta + \Delta$

$H' = H - h$

Δ **=** Offset in Declination due to mount misalignment

$\boldsymbol{h}$ **=** Offset in Hour Angle due to mount misalignment

We want to express Δ and $\boldsymbol{h}$ in terms of the mount axis errors θ and γ for specific values of δ and $\boldsymbol{H}$.

Start with a perfectly aligned telescope with Direction Cosines as in Eq 1-1 and rotate $\phi - \pi/2$ counterclockwise around the Y-axis to bring the Z-axis vertical, where ϕ is the observer's latitude. Then rotate around the Z-axis by an amount θ equal to the desired azimuth error in the Alt/Az reference frame. Then rotate clockwise around the new Y-axis by an amount $\pi/2 - \phi + \gamma$, where γ is the altitude error, also in the Alt/Az reference frame.

Note also:

$$\cos(\pi/2 - \Phi) = \cos(\Phi - \pi/2) = \sin(\Phi)$$
$$\sin(\pi/2 - \Phi) = \cos(\Phi)$$
$$\sin(\Phi - \pi/2) = -\cos(\Phi)$$

$$\begin{pmatrix} L_e' \\ M_e' \\ N_e' \end{pmatrix} = \begin{pmatrix} \sin(\Phi+\gamma) & 0 & \cos(\Phi+\gamma) \\ 0 & 1 & 0 \\ -\cos(\Phi+\gamma) & 0 & \sin(\Phi+\gamma) \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin\Phi & 0 & -\cos\Phi \\ 0 & 1 & 0 \\ \cos\Phi & 0 & \sin\Phi \end{pmatrix} \begin{pmatrix} L_e \\ M_e \\ N_e \end{pmatrix} \quad \text{Eq 1-3}$$

$$= \begin{pmatrix} \sin(\Phi+\gamma)\cos\theta & \sin(\Phi+\gamma)\sin\theta & \cos(\Phi+\gamma) \\ -\sin\theta & \cos\theta & 0 \\ -\cos(\Phi+\gamma)\cos\theta & -\cos(\Phi+\gamma)\sin\theta & \sin(\Phi+\gamma) \end{pmatrix} \begin{pmatrix} \sin\Phi & 0 & -\cos\Phi \\ 0 & 1 & 0 \\ \cos\Phi & 0 & \sin\Phi \end{pmatrix} \begin{pmatrix} L_e \\ M_e \\ N_e \end{pmatrix} \quad \text{Eq 1-4}$$

$$= \begin{pmatrix} \sin(\Phi+\gamma)\cos\theta\sin\Phi + \cos(\Phi+\gamma)\cos\Phi & \sin(\Phi+\gamma)\sin\theta & -\sin(\Phi+\gamma)\cos\theta\cos\Phi + \cos(\Phi+\gamma)\sin\Phi \\ -\sin\theta\sin\Phi & \cos\theta & \sin\theta\cos\Phi \\ -\cos(\Phi+\gamma)\cos\theta\sin\Phi + \sin(\Phi+\gamma)\cos\Phi & -\cos(\Phi+\gamma)\sin\theta & \cos(\Phi+\gamma)\cos\theta\cos\Phi + \sin(\Phi+\gamma)\sin\Phi \end{pmatrix} \begin{pmatrix} L_e \\ M_e \\ N_e \end{pmatrix} \quad \text{Eq 1-5}$$

Offset in Declination:

Consider N_e':

$$\begin{aligned} \sin\delta' = {} & -\cos\Phi\cos\gamma\cos\theta\sin\Phi\cos\delta\cos(-H) + \sin\Phi\sin\gamma\cos\theta\sin\Phi\cos\delta\cos(-H) \\ & + \sin\Phi\cos\gamma\cos\Phi\cos\delta\cos(-H) + \cos\Phi\sin\gamma\cos\Phi\cos\delta\cos(-H) \\ & - \cos\Phi\cos\gamma\sin\theta\cos\delta\sin(-H) + \sin\Phi\sin\gamma\sin\theta\cos\delta\sin(-H) \\ & + \cos\Phi\cos\gamma\cos\theta\cos\Phi\sin\delta - \sin\Phi\sin\gamma\cos\theta\cos\Phi\sin\delta \\ & + \sin\Phi\cos\gamma\sin\Phi\sin\delta + \cos\Phi\sin\gamma\sin\Phi\sin\delta \end{aligned} \quad \text{…Eq 1-6}$$

Assuming γ and θ are small:

$$\sin\gamma = \gamma \quad \cos\gamma = 1 \quad \sin\theta = \theta \quad \cos\theta = 1$$

$$\begin{aligned}
= & -\cos\Phi\sin\Phi\cos\delta\cos(-H) + \gamma\sin^2\Phi\cos\delta\cos(-H) \\
& + \sin\Phi\cos\Phi\cos\delta\cos(-H) + \gamma\cos^2\Phi\cos\delta\cos(-H) \\
& - \theta\cos\Phi\cos\delta\sin(-H) + \gamma\theta\sin\Phi\cos\delta\sin(-H) \\
& + \cos^2\Phi\sin\delta - \gamma\sin\Phi\cos\Phi\sin\delta \\
& + \sin^2\Phi\sin\delta + \gamma\cos\Phi\sin\Phi\sin\delta
\end{aligned}$$

…Eq 1-7

Note

$$\sin\delta' = \sin(\delta + \Delta) = \sin\delta\cos\Delta + \cos\delta\sin\Delta$$

$$= \sin\delta + \Delta\cos\delta$$

$$\Delta = \gamma\cos(-H) + \theta\sin(-H)(\cos\Phi + \gamma\sin\Phi)$$

…Eq 1-8

We can probably ignore the $\gamma\sin\Phi$ term, which involves the product of the two error terms.

Offset in Hour Angle:

Consider M_e':

$$\cos\delta'\sin(-H') = -\sin\theta\sin\Phi\cos\delta\cos(-H) + \cos\theta\cos\delta\sin(-H) + \sin\theta\cos\Phi\sin\delta$$

…Eq 1-9

Assuming θ is small, we have:

$$= -\theta\sin\Phi\cos\delta\cos(-H) + \cos\delta\sin(-H) + \theta\cos\Phi\sin\delta$$

…Eq 1-10

Consider L_e':

$$\cos\delta'\cos(-H') = \sin\Phi\cos\gamma\cos\theta\sin\Phi\cos\delta\cos(-H)$$
$$+ \cos\Phi\sin\gamma\cos\theta\sin\Phi\cos\delta\cos(-H)$$
$$+ \cos\Phi\cos\gamma\cos\Phi\cos\delta\cos(-H) - \sin\Phi\sin\gamma\cos\Phi\cos\delta\cos(-H)$$
$$+ \sin\Phi\cos\gamma\sin\theta\cos\delta\sin(-H) + \cos\Phi\sin\gamma\sin\theta\cos\delta\sin(-H)$$
$$- \sin\Phi\cos\gamma\cos\theta\cos\Phi\sin\delta - \cos\Phi\sin\gamma\cos\theta\cos\Phi\sin\delta$$
$$+ \cos\Phi\cos\gamma\sin\Phi\sin\delta - \sin\Phi\sin\gamma\sin\Phi\sin\delta \quad \text{...Eq 1-11}$$

Assuming θ and γ are small and reducing, we have:

$$= \theta\cos\delta\sin(-H)(\sin\Phi + \gamma\cos\Phi) - \gamma\sin\delta + \cos\delta\cos(H) \quad \text{...Eq 1-12}$$

The Hour Angle can be expressed in terms of the Direction Cosines as follows:

$$\tan(-H') = \frac{\cos\delta'\sin(-H')}{\cos\delta'\cos(-H')}$$

h *= Offset in Hour Angle*

$$= \frac{\sin(-H + h)}{\cos(-H + h)}$$

$$= \frac{\sin(-H)\cosh + \cos(-H)\sinh}{\cos(-H)\cosh - \sin(-H)\sinh}$$

h *assumed to be small.*

$$= \frac{\sin(-H) + h\cos(-H)}{\cos(-H) - h\sin(-H)}$$

$$\frac{\sin(-H) + h\cos(-H)}{\cos(-H) - h\sin(-H)} = \frac{-\theta\sin\Phi\cos\delta\cos(-H) + \cos\delta\sin(-H) + \theta\cos\Phi\sin\delta}{\theta\cos\delta\sin(-H)(\sin\Phi + \gamma\cos\Phi) - \gamma\sin\delta + \cos\delta\cos(H)} \quad \text{…Eq 1-13}$$

$$\theta\cos\delta\sin^2(-H)(\sin\Phi + \gamma\cos\Phi) - \gamma\sin(-H)\sin\delta + \cos\delta\cos(H)\sin(-H)$$
$$+ h\theta\cos\delta\sin(-H)\cos(-H)(\sin\Phi + \gamma\cos\Phi) - h\gamma\cos(-H)\sin\delta + h\cos^2(-H)\cos\delta$$

$$= -\theta\sin\Phi\cos\delta\cos^2(-H) + \cos\delta\sin(-H)\cos(-H) + \theta\cos\Phi\cos(-H)\sin\delta$$
$$+ h\theta\sin\Phi\cos\delta\cos(-H)\sin(-H) - h\sin^2(-H)\cos\delta - h\theta\cos\Phi\sin(-H)\sin\delta \quad \text{…Eq 1-14}$$

$$h\theta\cos\delta\sin(-H)\cos(-H)(\sin\Phi + \gamma\cos\Phi) - h\gamma\cos(-H)\sin\delta + h\cos^2(-H)\cos\delta$$
$$- h\theta\sin\Phi\cos\delta\cos(-H)\sin(-H) + h\sin^2(-H)\cos\delta + h\theta\cos\Phi\sin(-H)\sin\delta$$

$$= -\theta\sin\Phi\cos\delta\cos^2(-H) + \cos\delta\sin(-H)\cos(-H) + \theta\cos\Phi\cos(-H)\sin\delta$$
$$- \theta\cos\delta\sin^2(-H)(\sin\Phi + \gamma\cos\Phi) + \gamma\sin(-H)\sin\delta - \cos\delta\cos(H)\sin(-H) \quad \text{…Eq 1-15}$$

$$h(-\gamma\cos(-H)\sin\delta + \theta\cos\Phi\sin(-H)\sin\delta + \cos\delta)$$
$$= -\theta\sin\Phi\cos\delta + \theta\cos\Phi\cos(-H)\sin\delta + \gamma\sin(-H)\sin\delta \quad \text{…Eq 1-16}$$

Assume all terms in **h**γ and **h**θ are small and solving for **h**, we have:

$$h = \gamma\sin(-H)\tan\delta + \theta(\cos\Phi\cos(-H)\tan\delta - \sin\Phi) \quad \text{…Eq 1-17}$$

The two Basic Equations governing the change in declination and Hour Angle for small changes in mount elevation and azimuth become:

$$\Delta = \gamma\cos\eta + \theta\cos\Phi\sin\eta \qquad \text{...Eq 1-18}$$

$$h = \gamma\tan\delta\sin\eta + \theta(\cos\Phi\tan\delta\cos\eta - \sin\Phi) \qquad \text{...Eq 1-19}$$

where:

Δ = Offset in Declination
h = Offset in Hour Angle
γ = Polar Axis Error in Elevation
θ = Polar Axis Error in Azimuth
δ = Declination of reference star
η = ***-H*** = Hour Angle of reference star measured from South to West
Φ = Local Latitude

Measuring Polar Axis Alignment Errors:

We can measure the polar axis alignment errors by moving between a pair of known stars and noting the offset in Hour Angle and Declination after the movement. The mount errors can be expressed in matrix format.

From the above Basic Equations, we have:

$$\begin{bmatrix} h \\ \Delta \end{bmatrix} = \begin{bmatrix} \tan\delta_2\sin\eta_2 - \tan\delta_1\sin\eta_1 & \cos\Phi(\tan\delta_2\cos\eta_2 - \tan\delta_1\cos\eta_1) \\ \cos\eta_2 - \cos\eta_1 & \cos\Phi(\sin\eta_2 - \sin\eta_1) \end{bmatrix} \begin{bmatrix} \gamma \\ \theta \end{bmatrix} \qquad \text{...Eq 1-20}$$

Taking the inverse of the 2 X 2 matrix, we have:

$$\begin{bmatrix} \gamma \\ \theta \end{bmatrix} = \begin{bmatrix} \cos\Phi(\sin\eta_2 - \sin\eta_1)/D & \cos\Phi(\tan\delta_1\cos\eta_1 - \tan\delta_2\cos\eta_2)/D \\ (\cos\eta_1 - \cos\eta_2)/D & (\tan\delta_2\sin\eta_2 - \tan\delta_1\sin\eta_1)/D \end{bmatrix} \begin{bmatrix} h \\ \Delta \end{bmatrix}$$

...Eq 1-21

where ***D*** is the determinant of the original matrix

$$D = \cos\Phi[(\tan\delta_1 + \tan\delta_2)\cos(\eta_1 + \eta_2) - \tan\delta_1\cos2\eta_1 - \tan\delta_2\cos2\eta_2]$$

Having determined the mount errors using Eq 1-21, the values can be used in Eq 1-18 and Eq 1-19 to calculate an offset from a star such that re-centering the star will result in aligning the polar axis.

Southern Hemisphere considerations:

Equation 1-18 has a term in $\sin\Phi$ that will change sign in the Southern Hemisphere. However, as the typical setup is to mount the telescope with the polar axis pointing South rather than North we should be able to use the absolute value of the sine term, treating the Southern Hemisphere as a mirror image of the Northern Hemisphere.

Choice of suitable stars:

Stars should be at least 15 degrees above the horizon to reduce the effects of atmospheric refraction. They should also be separated by at least 15 degrees in both declination and Right Ascension. Best results are achieved when the separation in declination and Right Ascension are similar.

Stars must be on the same side of the meridian.

Stars equidistant above and below the celestial equator should be avoided because this will drive terms in the determinant towards zero, making calculations prone to rounding errors.

Stars close to the zenith or due East or West should also be avoided because their apparent position is insensitive to changes in polar axis azimuth or elevation.

Sources of Error:

The entire procedure is based upon the telescope's ability to accurately move from one object to another. Motor errors, mount fabrication errors, (non-orthogonal axes), backlash, etc. will negatively impact a telescope's positional accuracy. Polar axis alignment therefore will be limited by the telescope's overall accuracy. Fortunately, all of the above errors can be diagnosed and either corrected or accounted for.

During the time to move between Star 1 and Star 2 and the time taken to center the second star, the telescope will be tracking around an incorrect polar axis. This will cause the stars to drift in the field of view introducing additional errors that the above formulae don't account for. Indeed, making use of this drift is frequently used to polar align a telescope. However, drifts of 15 to 30 minutes are usually needed to make adequate adjustments while movement between a pair of stars and centering takes only a few minutes so this error is small. However, a second pass is usually required to get the axis errors down to acceptable levels.