Revisiting Some Analytical Solutions to Incompressible Flow Using Projection by Inspection

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Abstract

It has been previously asserted by this author that the well-known weak orthogonal decomposition of the Navier-Stokes equation into solenoidal and irrotational parts is also fundamental from a physical point of view. It was asserted that incompressible flow is governed by a pressureless integro-differential (momentum) equation, and the pressure by a complementary integro-differential functional of the velocity field. These equations can be formally written in terms of solenoidal and irrotational projection operators. In this view, the Navier-Stokes equation is merely the composition of these two more fundamental equations.

As the representation of these projection operators is difficult to deal with directly, other methods are desirable. In the weak formulation, projections are effected by the orthogonality of the solenoidal and irrotational test spaces.

This paper revisits some known closed-form or "analytical" solutions of the Navier-Stokes equation, and using "projection by inspection," shows how these satisfy the more-fundamental velocity and pressure equations. Thus, adoption of the fundamental complementary equations in place of the composite incompressible Navier-Stokes equations offers no additional mathematical burden to the teaching of elementary fluid dynamics.

1 Introduction

For 150 years the Navier-Stokes equation has been accepted as the canonical governing equation for incompressible flow. The notion of incompressible flow is of course an idealization, and is purported to describe the flow of a hypothetical "incompressible fluid" and to be representative of the flow of compressible fluids at velocities well below the velocity of propagation of pressure disturbances.

From the engineering literature, it is apparent that the nature of the Navier-Stokes equation is not generally well-understood in the context of numerical computations. This is evident from the frequently-repeated concept that "the pressure enforces incompressibility" in the flows.

While mathematicians since J. Leray [7] have understood the possibility and utility of orthogonally decomposing the Navier-stokes equation into a pressureless governing equation for incompressible flow and an equation for the pressure (gradient) as a functional of the velocity field, none seem to have asserted the decomposition as fundamental from a physical point of view.

Consider the classical derivation of the equation of motion of an incompressible fluid from the physical perspective. One considers the balance of momentum of the fluid in a small volume $\Delta V = \Delta x \Delta y \Delta z$. By Newton's second law, the change in momentum per unit time of the fluid in the volume is equal to that transfered across the boundary plus the sum of the forces acting on the enclosed fluid. It is argued that one of the forces results from a difference in force across opposite faces from a pressure difference on the faces. The difference in force per unit volume on the two faces normal to the x-axis is claimed to be

$$-\frac{(p_2 - p_1)\Delta y \Delta z}{\Delta x \Delta y \Delta z} = -\frac{\Delta p}{\Delta x} \sim -\nabla p. \tag{1}$$

However, since pressure disturbances propagate at an infinite velocity in an incompressible medium, a pressure differential across the volume would be instantly equilibrated. As a dynamic pressure difference cannot exist for finite time, there can be no pressure gradient term in the equation of motion for an incompressible fluid. Likewise, only the nonconservative body forces can affect the change in momentum [3][4]. However, since functions of a solenoidal field might not be solenoidal, the equation of momentum of an incompressible fluid is given formally by,

$$\frac{\partial}{\partial t}\mathbf{u} = \pi^{\mathrm{S}}(-\mathbf{u}\cdot\nabla\mathbf{u} + \nu\nabla^2\mathbf{u}) + \mathbf{f}^{\mathrm{S}}, \tag{2}$$

where π^{S} is an operator which projects out the solenoidal part of its operand.

The effective or complementary pressure is given as a function of the flow and any conservative body forces by (an algebraic equation in time),

$$\nabla p = \pi^{\mathrm{I}}(-\mathbf{u}\cdot\nabla\mathbf{u} + \nu\nabla^2\mathbf{u}) + \mathbf{f}^{\mathrm{I}}, \tag{3}$$

where the operator π^{I} projects out the irrotational part of its operand. These operators are given formally by [1],

$$\pi^{\mathrm{I}} = \nabla\Delta^{-1}\nabla\cdot\ , \quad \pi^{\mathrm{S}} = 1 - \pi^{\mathrm{I}}, \tag{4}$$

where the operator Δ^{-1} (the inverse Laplacian) is the Green's function for the Laplace equation on the problem domain Ω. The complementary pair of equations are then,

$$\begin{aligned} \tfrac{\partial}{\partial t}\mathbf{u} &= \pi^{\mathrm{S}}(-\mathbf{u}\cdot\nabla\mathbf{u} + \nu\nabla^2\mathbf{u}) + \mathbf{f}^{\mathrm{S}}, \\ \nabla p &= \pi^{\mathrm{I}}\,(-\mathbf{u}\cdot\nabla\mathbf{u} + \nu\nabla^2\mathbf{u}) + \mathbf{f}^{\mathrm{I}}, \end{aligned} \tag{5}$$

When these two equations are combined noting $\pi^{\mathrm{S}} + \pi^{\mathrm{I}} = 1$, the result is the Navier-Stokes equation,

$$\frac{\partial}{\partial t}\mathbf{u} = -\mathbf{u}\cdot\nabla\mathbf{u} + \nu\nabla^2\mathbf{u} - \nabla p + \mathbf{f}. \tag{6}$$

The Navier-Stokes equation seems mathematically simpler as it does not contain the projection operators. It is, however, a differential algebraic equation (ADE), which poses problems of its own when advancing the flow in time.

The equations above are indeterminate. For any given problem one must specify the problem domain Ω and the velocity on Dirichlet portions of the boundary. For steady, bounded problems with flow-through portions of the boundary or domains bounded in all but one coordinate direction, the flux,

$$\phi = \int_S \mathbf{u}\cdot\mathbf{n}dS, \tag{7}$$

through S must be specified or determined, where S is a disjoint flow-through boundary or the intersection of the domain with a plane. This or equivalent data is necessary, but may not be sufficient, for the solution to exist.

In two dimensions the integral is just the difference in stream function across such an aperture, $\phi = \Delta\psi$. In three dimensions the integral is equal to the of the tangential component of the vector potential $\mathbf{A}$ around the aperture,

$$\phi = \oint \mathbf{A}\cdot\tau d\tau. \tag{8}$$

Thus, specification of boundary conditions on the stream function or vector potential might be equivalent to the specification of fluxes.

2 Some Simple Examples

It is useful to revisit some known analytical solutions of incompressible flow in the context of this formalism. The technique to be used here might be called "projection by inspection." The standard derivations in terms of the Navier-Stokes equation can be found in many introductory textbooks (e.g. White[11], Pozrikidis[8]).

F. G. White [11] remarks that there are two types of exact solutions, linear solutions where the convective acceleration $\mathbf{u} \cdot \nabla \mathbf{u}$ vanishes, and nonlinear solutions where this term does not vanish. He also classifies solutions by type or geometry of flow involved: (1) Couette, (2) steady duct flows, (3) unsteady duct flows, (4) flows with moving boundaries, (5) asymptotic suction flows, (6) wind-driven Ekman flows, and (7) similarity solutions (rotating disk, stagnation flow, wedge flow). A number of these cases are explored in examples to follow.

It is useful to recognize some nonconservative body forces. The first is associated with electromagnetically-pumped channel flow. The fluid is assumed to be electrically conducting (e.g. saltwater, liquid mercury, or molten metal). The walls are electrically conducting and an electrical potential is applied, resulting in an electrical current density $\mathbf{j}$ across the channel. This $\mathbf{j}$ can be related to the electrical conductivity of the fluid and the voltage applied across the channel. A magnetic induction field $\mathbf{B}$ is applied perpendicular to the plane of the channel. This results in a body force of magnitude $f^S = |\mathbf{j} \times \mathbf{B}| = jB$ along the channel. Joule heating effects are neglected.

A second force is associated with thermally-driven flow. While the density of an incompressible fluid is independent of pressure, it is assumed to vary slightly with temperature as $\rho = \rho_0(1 + \beta(T - T_0)$, giving rise to a buoyant force in the vertical direction of magnitude $f^S = -g(\rho - \rho_0) = -g\beta\Delta T$, where g is the gravitational acceleration and β is the volumetric thermal expansion coefficient. When the variation of density in all other terms is neglected, this is referred to as the Boussinesque approximation.

Driven Channel Flow. For the first example, consider the internal two-dimensional problem domain bounded by $y = \pm 1$, perhaps driven by a nonconservative force directed along the channel, or with a consistent pressure gradient, and perhaps with specified flow $\phi = \int_S \mathbf{u} \cdot \mathbf{n} dS$, where S is the intersection of the problem domain with a surface through the domain. The simplest steady polynomial velocity function vanishing on the channel boundary is,

$$\mathbf{u} = \begin{bmatrix} V_0(1 - y^2) \\ 0 \end{bmatrix}, \tag{9}$$

with the corresponding stream function $\psi = V_0 y(1 - y^2/3)$. This trial solution is associated with fully-developed flow in a channel. By simple computation, we find the trivial decompositions,

$$\nu\nabla^2\mathbf{u} = \begin{bmatrix} -\alpha 2\nu V_0 \\ 0 \end{bmatrix} + \begin{bmatrix} -(1-\alpha)2\nu V_0 \\ 0 \end{bmatrix}, \qquad \mathbf{u} \cdot \nabla \mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{10}$$

where the first term on the right side of the diffusion is assumed to be solenoidal and the second irrotational. The arbitrariness in the form of the diffusion term reflects the fact that it could derive from a curl or a gradient, i.e. it could be solenoidal or irrotational. The choice will be determined by the equations. Applying (5),

$$\begin{aligned} \pi^{\mathrm{S}}(-\mathbf{u} \cdot \nabla \mathbf{u} + \nu\nabla^2\mathbf{u}) &= \begin{bmatrix} -\alpha 2\nu V_0 \\ 0 \end{bmatrix} = -\mathbf{f}^S\,, \\ \pi^{\mathrm{I}}(-\mathbf{u} \cdot \nabla \mathbf{u} + \nu\nabla^2\mathbf{u}) &= \begin{bmatrix} -(1-\alpha)2\nu V_0 \\ 0 \end{bmatrix} = \nabla p\,. \end{aligned} \tag{11}$$

If the pressure gradient $\nabla p = 0$ and ϕ is not specified, then $\alpha = 1$ and $2\nu V_0 = \mathrm{f}^S$. For electromagnetically-pumped flow, the expression for the electromagnetic force $|\mathbf{j} \times \mathbf{B}|$ is substituted. If ϕ and f^S were specified and not ∇p, this could lead to an additional pressure gradient.

If the nonconservative driving force $\mathbf{f}^S = 0$, requiring the residual of the momentum term to vanish gives $\alpha = 0$, from which $\partial p/\partial x = -2\nu V_0$. From this it is usual to assert that the magnitude of the flow depends on (or is consistent with) the pressure gradient. In this case, the pressure gradient derives from the residual of the diffusion term.

In light of the relation between the flow and the stream function and the fact that the equation of fluid motion does not depend directly on the pressure field, the flow could more properly be expressed in terms of a boundary condition, the difference of the stream function values across the channel, $\phi = \Delta\psi$. In this case $V_0 = \frac{4}{3}\phi$. Clearly these two views are related by $\partial p/\partial x = -3\nu\phi/2$.

Gravity-driven Channel Flow. This example uses a conservative (gravitational) body force. The flow domain is a two-dimensional channel between plane walls with nonslip boundary conditions. The channel is inclined at an angle $-\beta$ with respect to the horizontal. The coordinate system is also rotated through the angle $-\beta$. In this system, the walls are located at $y = \pm 1$ and $-\infty < x < \infty$. The complementary velocity-pressure equations for steady laminar flow are then,

$$\begin{aligned} 0 &= \pi^{\mathrm{S}}(-\mathbf{u}\cdot\nabla\mathbf{u} + \nu\nabla^2\mathbf{u}), \\ \nabla p &= \pi^{\mathrm{I}}(-\mathbf{u}\cdot\nabla\mathbf{u} + \nu\nabla^2\mathbf{u}) - \mathbf{g}, \end{aligned} \tag{12}$$

where $\mathbf{g}$, the gravitational acceleration, is the gradient of the gravitational potential.

The trial solution for the velocity is of the form,

$$\mathbf{u} = \begin{bmatrix} V_0(1-y^2) \\ 0 \end{bmatrix}. \tag{13}$$

Then we write the decompositions,

$$\nu\nabla^2\mathbf{u} = \begin{bmatrix} -\alpha 2\nu V_0 \\ 0 \end{bmatrix} + \begin{bmatrix} -(1-\alpha)2\nu V_0 \\ 0 \end{bmatrix}, \qquad \mathbf{u}\cdot\nabla\mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{14}$$

where the first (constant) term in the convection is assumed to be solenoidal and the second irrotational. Substituting into the complementary equations gives,

$$\begin{aligned} 0 &= \begin{bmatrix} -\alpha 2\nu V_0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \nabla p &= \begin{bmatrix} -(1-\alpha)2\nu V_0 \\ 0 \end{bmatrix} - \begin{bmatrix} g\sin\beta \\ g\cos\beta \end{bmatrix}, \end{aligned} \tag{15}$$

Requiring the residual in the velocity equation to vanish gives $\alpha = 0$. Then

$$\nabla p = \begin{bmatrix} -2\nu V_0 - g\sin\beta \\ -g\cos\beta \end{bmatrix} \tag{16}$$

Apart from an additional constant, the complementary pressure p is then,

$$p = (-2\nu V_0 - g\sin\beta)x - gy\cos\beta\,. \tag{17}$$

This gives a relation between the parameter V_0 in terms of the problem data. Specifying V_0 (or the net flow) determines the unique solution.

Non-conservative forces (which appear to be absent in this problem) drive the flow, but conservative forces can produce a similar effect by requiring that the flow be compatible with them. For steady flow to be just that for infinite time, there must be some means to keep the implied upstream reservoir supplying fluid full. This would require a nonconservative force (a pump), and this would be the driving force at a global level.

Steady Thermally-driven Flow in an Infinite Vertical Channel. Consider a channel parallel to the y-axis, bounded by two stationary infinite planes at $x = \pm h$. Let the temperature of the right plane be T_2 and the left T_1. As the steady flow is parallel to the planes, there is no convective heat transfer across the channel. The heat transfer is conductive without sources so the temperature distribution is $T = T_0 + x\Delta T/2h$ with the average temperature $T_0 = (T_1 + T_2)/2$. The resulting y-component of the nonconservative body force in the Boussinesque approximation is $f^S = -xg\beta\Delta T/2h$. We verify that the velocity is of the form

$$\mathbf{u} = \begin{bmatrix} 0 \\ A(1 - x^2/h^2)x/h \end{bmatrix}. \tag{18}$$

From the convective and diffusive terms we have,

$$\mathbf{u}\cdot\nabla\mathbf{u} = 0, \qquad \nu\nabla^2\mathbf{u} = \begin{bmatrix} 0 \\ 6\nu A x/h^3 \end{bmatrix}. \tag{19}$$

The diffusive term is regarded as solenoidal, so

$$\begin{aligned} \pi^{\mathrm{S}}(-\mathbf{u}\cdot\nabla\mathbf{u}+\nu\nabla^2\mathbf{u}+\mathbf{f}^S) &= \left[\begin{array}{c} 0 \\ 6\nu Ax/h^3 \end{array}\right] - \left[\begin{array}{c} 0 \\ g\beta\Delta Tx/2h \end{array}\right] = 0\,, \\ \pi^{\mathrm{I}}(-\mathbf{u}\cdot\nabla\mathbf{u}+\nu\nabla^2\mathbf{u}) &= \left[\begin{array}{c} 0 \\ 0 \end{array}\right] = \nabla p\,. \end{aligned} \tag{20}$$

The requirement that the residual of the first term vanish gives

$$A = \frac{g\beta h^2\Delta T}{12\nu}. \tag{21}$$

Boundary-driven Channel Flow. Consider steady two-dimensional incompressible flow through an infinite channel bounded by parallel walls separated a distance h. Let the walls be translating with velocities $u = U_1$ at $y = 0$ and $u = U_2$ at $y = h$. As this flow is translationally invariant, the velocity is parallel to the x-axis and is a function of y only. A polynomial trial function satisfying the no-slip boundary conditions is given by,

$$\mathbf{u} = \left[\begin{array}{c} U_1 y/h + U_2(1-y/h) + \gamma y/h(1-y/h) \\ 0 \end{array}\right]. \tag{22}$$

The first two terms constitute Couette flow and the third Poiseulle flow. Evaluating the diffusion and convection terms gives,

$$\nu\nabla^2\mathbf{u} = \left[\begin{array}{c} \alpha(2\nu\gamma/h^2) \\ 0 \end{array}\right] + \left[\begin{array}{c} (1-\alpha)(2\nu\gamma/h^2) \\ 0 \end{array}\right], \qquad \mathbf{u}\cdot\nabla\mathbf{u} = 0. \tag{23}$$

The first term in the (constant) diffusion is assumed to be the solenoidal part and the second the irrotational part. Substitution into the complementary $v-p$ equations gives,

$$\begin{aligned} 0 &= \left[\begin{array}{c} \alpha(2\nu\gamma/h^2) \\ 0 \end{array}\right], \\ \nabla p &= \left[\begin{array}{c} (1-\alpha)(2\nu\gamma/h^2) \\ 0 \end{array}\right]. \end{aligned} \tag{24}$$

Requiring the residual in the (first) velocity term to vanish gives $\alpha = 0$ and $\frac{\partial p}{\partial x} = 2\nu\gamma/h^2$. Specification of a pressure gradient or a net flow gives a unique solution.

Circular Couette Flow. We consider steady laminar viscous flow between two rotating cylinders. Let the inner cylinder radius and angular velocity be R_1 and Ω_1, and the outer cylinder radius and angular velocity be R_2 and Ω_2, respectively. Assuming the flow to be rotationally invariant, it is a function of the radius r only. Assuming the flow to be steady, the radial component of the flow must vanish. Using polar coordinates,the flow must then be of the form,

$$u_r = 0, \qquad u_\theta = ar + br^{-1}. \tag{25}$$

Applying the no-slip boundary conditions, we evaluate the parameters above to be,

$$a = \frac{\Omega_2 - \alpha\Omega_1}{1-\alpha}, \qquad b = -\frac{\Omega_2 - \Omega_1}{1-\alpha}, \tag{26}$$

where $\alpha = R_1^2/R_2^2$. Evaluating the convection and diffusion terms and projecting into solenoidal and a rotational parts respectively, we find,

$$\nu\nabla^2\mathbf{u} = 0, \qquad \mathbf{u}\cdot\nabla\mathbf{u} = \left[\begin{array}{c} 0 \\ 0 \end{array}\right] + \left[\begin{array}{c} -r^{-1}u_\theta^2 \\ 0 \end{array}\right]. \tag{27}$$

Substitution into the complementary equations gives,

$$\begin{aligned} 0 &= \left[\begin{array}{c} 0 \\ 0 \end{array}\right], \\ \nabla p &= \left[\begin{array}{c} -r^{-1}u_\theta^2 \\ 0 \end{array}\right]. \end{aligned} \tag{28}$$

Thus,

$$
\begin{aligned}
u_r &= 0, & u_\theta &= \frac{\Omega_2 - \alpha\Omega_1}{1-\alpha} r - \frac{\Omega_2 - \Omega_1}{1-\alpha} r^{-1}, \\
\partial p/\partial r &= -r^{-1} u_\theta^2, & \partial p/\partial \theta &= 0.
\end{aligned} \tag{29}
$$

The third equation is easily integrated to find the explicit form for the pressure. In this problem the pressure derives from the momentum term.

3 Parallel Oscillating Flow

These problems involve periodic laminar motion of a viscous incompressible fluid, parallel to an infinite bounding plane or planes. The boundaries may be in periodic motion or the fluid may be subject to a periodic body force. This tends to generate a virtual transverse wave near the boundary. As a fluid cannot support a propagating transverse wave, this wave is exponentially damped, forming a boundary layer. As the flow velocity is a function of the distance from the boundary only, the convection term $\mathbf{u} \cdot \nabla \mathbf{u}$ vanishes.

A Simple Example. Consider a fluid filling the entire plane moving with velocity

$$
\mathbf{u} = \begin{bmatrix} U \sin \Omega t \\ 0 \end{bmatrix} \tag{30}
$$

under the influence of nonconservative body force $\mathbf{f}^S$ or perhaps a pressure gradient ∇p. Then

$$
\frac{\partial \mathbf{u}}{\partial t} = \begin{bmatrix} \alpha \Omega U \cos \Omega t \\ 0 \end{bmatrix} + \begin{bmatrix} (1-\alpha)\Omega U \cos \Omega t \\ 0 \end{bmatrix}, \tag{31}
$$

where the first term is assumed to be solenoidal and the second irrotational and α is a parameter to be determined. This form is used because the term cannot be determined to be solenoidal or irrotational by inspection in isolation. We then write,

$$
\begin{aligned}
0 &= -\frac{\partial \mathbf{u}}{\partial t} + \pi^{\mathrm{S}}(-\mathbf{u}\cdot\nabla\mathbf{u} + \nu\nabla^2\mathbf{u}) + \mathbf{f}^S = \begin{bmatrix} -\alpha\Omega U \cos\Omega t + \mathbf{f}^S \\ 0 \end{bmatrix}, \\
\nabla p &= \pi^{\mathrm{I}}(-\mathbf{u}\cdot\nabla\mathbf{u} + \nu\nabla^2\mathbf{u} = \begin{bmatrix} (1-\alpha)\Omega U \cos\Omega t) \\ 0 \end{bmatrix}.
\end{aligned} \tag{32}
$$

There are then two cases. Taking $\alpha = 1$, $\nabla p = 0$, and the nonconservative force $\mathbf{f}^S = \Omega U \cos \Omega t$. Taking $\alpha = 0$, $\mathbf{f}^S = 0$ and $\nabla p = \Omega U \cos \Omega t$.

Semi-infinite Flow Above an Oscillating Plane. Consider the laminar viscous flow generated by an infinite flat plate oscillating in its plane along the x-axis. Let the velocity of this plane be given by $U \cos \Omega t$. As the flow is invariant under translation along the x-axis, the flow is parallel to the x-axis and is a function of the y-coordinate only. The fluid is assumed to be at rest at $y = +\infty$ with no slippage at the plane, where $u(y = 0, t) = U \cos \Omega t$. This generates flow transverse to the normal of the plane. As a transverse wave cannot propagate in a fluid, this motion at the plane is damped and rapidly decays to zero at $+\infty$. This suggests a trial solution of the form

$$
\mathbf{u} = \begin{bmatrix} U \exp(-\beta y)\cos(\Omega t - ky) \\ 0 \end{bmatrix}. \tag{33}
$$

Then,

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \begin{bmatrix} -U\Omega \exp(-\beta y)\sin(\Omega t - ky) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \\
\nu\nabla^2\mathbf{u} &= \begin{bmatrix} -\nu U \exp(-\beta y)\{(\beta^2 - k^2)\cos(\Omega t - ky) - 2\beta k \sin(\Omega t - ky)\} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\mathbf{u}\cdot\nabla\mathbf{u} &= \mathbf{0},
\end{aligned} \tag{34}
$$

where, by inspection, the first term in each expression is solenoidal and the second is irrotational. Consequently,

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} &= \pi^{\mathrm{S}}(-\mathbf{u}\cdot\nabla\mathbf{u}+\nu\nabla^2\mathbf{u}) = \begin{bmatrix} -U\Omega\exp(-\beta y)\sin(\Omega t-ky) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\nu U\exp(-\beta y)\{(\beta^2-k^2)\cos(\Omega t-ky)-2\beta k\sin(\Omega t-ky)\} \\ 0 \end{bmatrix}, \\ \nabla p &= \pi^{\mathrm{I}}(-\mathbf{u}\cdot\nabla\mathbf{u}+\nu\nabla^2\mathbf{u}) = \mathbf{0}\,.\end{aligned} \tag{35}$$

For the first relation to hold, $\beta^2 = k^2$, $2\nu\beta k = \Omega$ and $\beta = k = (\Omega/2\nu)^{\frac{1}{2}}$, so

$$\begin{aligned}\mathbf{u} &= \begin{bmatrix} U\exp(-(\Omega/2\nu)^{\frac{1}{2}}y)\cos(\Omega t-(\Omega/2\nu)^{\frac{1}{2}}y) \\ 0 \end{bmatrix}, \\ p &= \text{constant}\,.\end{aligned} \tag{36}$$

The non-propagating wave is bound to the generating surface in a layer with thickness $\approx (2\nu/\Omega)^{\frac{1}{2}}$.

Semi-infinite Oscillating Flow Above a Stationary Plane. This problem assumes a viscous fluid in the domain above a stationary plane. The fluid is oscillating parallel to the plane in response to a driving force, which may be an oscillating pressure gradient or a nonconservative body force.

Now consider the semi-infinite domain above a stationary plane with normal in the y-direction. This problem is complementary to the preceding one, and we will confirm that the velocity is given by

$$\mathbf{u} = \begin{bmatrix} U(\cos(\Omega t)-\exp(-\beta y)\cos(\Omega t-ky)) \\ 0 \end{bmatrix}. \tag{37}$$

with a nonconservative body force is given by,

$$\mathbf{f}^S = \hat{\mathbf{x}}A\sin\Omega t. \tag{38}$$

Then the decomposition of the diffusion and convection terms is,

$$\begin{aligned}\frac{\partial u}{\partial t} &= \begin{bmatrix} -U\Omega(\sin\Omega t-\exp(-\beta y)\sin(\Omega t-ky)) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \\ \nu\nabla^2\mathbf{u} &= \begin{bmatrix} -\nu U\exp(-\beta y)\{(\beta^2-k^2)\cos(\Omega t-ky)-2\beta k\sin(\Omega t-ky)\} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \mathbf{u}\cdot\nabla\mathbf{u} &= \mathbf{0}\,,\end{aligned} \tag{39}$$

with decomposition by inspection, where the first term in the diffusion is solenoidal and the second is irrotational. Consequently,

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} &= \pi^{\mathrm{S}}(-\mathbf{u}\cdot\nabla\mathbf{u}+\nu\nabla^2\mathbf{u}) = \begin{bmatrix} -U\Omega(\sin(\Omega t)-\exp(-\beta y)\sin(\Omega t-ky)) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\nu U\exp(-\beta y)\{(\beta^2-k^2)\cos(\Omega t-ky)-2\beta k\sin(\Omega t-ky)\}+A\sin(\Omega t) \\ 0 \end{bmatrix}, \\ \nabla p &= \pi^{\mathrm{I}}(-\mathbf{u}\cdot\nabla\mathbf{u}+\nu\nabla^2\mathbf{u}) = \mathbf{0}\,.\end{aligned} \tag{40}$$

For the first relation to hold, $\beta^2 = k^2$, $2\nu\beta k = \Omega$ and $\beta = k = (\Omega/2\nu)^{\frac{1}{2}}$ and $U = A/\Omega$, so

$$\begin{aligned}\mathbf{u} &= \begin{bmatrix} A/\Omega(\cos(\Omega t-(\Omega/2\nu)^{\frac{1}{2}}y)\exp(-(\Omega/2\nu)^{\frac{1}{2}}y)\cos(\Omega t-(\Omega/2\nu)^{\frac{1}{2}}y)) \\ 0 \end{bmatrix}, \\ p &= \text{constant}\,.\end{aligned} \tag{41}$$

The velocity is the standard result quoted for this problem, but the pressure is quite different.

Oscillatory Poiseulle Flow. Consider the flow in a channel with straight sides at $y = \pm h/2$ due to an oscillatory pressure gradient of the form $A \sin \Omega t$. The flow is expected to be parallel to and vanishing on the boundary. We verify the solution is the real part of the form,

$$u = -\frac{A}{\Omega} e^{-i\Omega t} \left(1 - \frac{\cosh[(-i\omega/\nu)^{\frac{1}{2}}(y - h/2)]}{\cosh[(-i\omega/\nu)^{\frac{1}{2}} h/2]} \right). \tag{42}$$

The convection term $\mathbf{u} \cdot \nabla \mathbf{u}$ vanishes, and the remaining terms in the complementary equations are the real parts of,

$$\nu \nabla^2 u = i A e^{-i\Omega t} \left(\frac{\cosh[(-i\omega/\nu)^{\frac{1}{2}}(y - h/2)]}{\cosh[(-i\omega/\nu)^{\frac{1}{2}} h/2]} \right) \tag{43}$$

and

$$\frac{\partial u}{\partial t} = A \sin \Omega t + i A e^{-i\Omega t} \left(\frac{\cosh[(-i\omega/\nu)^{\frac{1}{2}}(y - h/2)]}{\cosh[(-i\omega/\nu)^{\frac{1}{2}} h/2]} \right) \tag{44}$$

Then,

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \pi^{\mathrm{S}}(-\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u}) &= \begin{bmatrix} \alpha A \sin \Omega t \\ 0 \end{bmatrix} = \mathbf{0}\,, \\ \pi^{\mathrm{I}}(-\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u}) &= \begin{bmatrix} (1-\alpha) A \sin \Omega t \\ 0 \end{bmatrix} = \nabla p\,. \end{aligned} \tag{45}$$

We require that the residual of the velocity equation vanish yielding $\alpha = 0$. Then the pressure gradient is of the form required.

Transient Translation of a Plane. Consider the semi-infinite flow of an initially stationary fluid above a flat plate. The plate is suddenly set in motion with the constant velocity U, so the flow is parallel to the plane. We solve the equations (5)with the initial condition $u(y, t = 0) = 0$ and boundary conditions $u(y = 0, t > 0) = U$ and $u(y = \infty, t) = 0$.

We expect the solution will be proportional to U, so we write $u = UF(y, t, \nu)$, where F is dimensionless. As there are no external length or time scales, we look for a solution in terms of a dimensionless variable $\eta = y/(\nu t)^{\frac{1}{2}}$. The boundary conditions on F are $F(0) = 1$ and $F(\infty) = 0$. Then,

$$\begin{aligned} \frac{\partial F}{\partial t} &= U \frac{\partial F}{\partial \eta} \frac{\partial \eta}{\partial t}, \\ \nu \nabla^2 u &= \nu U \frac{\partial \eta}{\partial y} \frac{\partial}{\partial y} \frac{\partial F}{\partial \eta}, \\ \mathbf{u} \cdot \nabla \mathbf{u} &= \mathbf{0}. \end{aligned} \tag{46}$$

where the terms are solenoidal. From (5) it follows that,

$$\begin{aligned} \frac{\partial \eta}{\partial t} \frac{\partial F}{\partial \eta} &= \nu \left(\frac{\partial \eta}{\partial y} \right)^2 \frac{\partial}{\partial \eta} \frac{\partial F}{\partial \eta} \\ \nabla p &= 0. \end{aligned} \tag{47}$$

Then,

$$-\frac{1}{2} \eta \frac{\partial F}{\partial \eta} = \frac{\partial}{\partial \eta} \left(\frac{\partial F}{\partial \eta} \right). \tag{48}$$

Integrating once gives,

$$\frac{\partial F}{\partial \eta} = C_1 e^{-\eta^2/4} + C_2. \tag{49}$$

Integrating again,

$$F = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta/2} e^{-z^2} dz = 1 - \mathrm{erf}(\eta/2), \tag{50}$$

where we have applied the boundary conditions on F, and of course the pressure is constant and may be taken to be zero. This represents a boundary layer which grows with a velocity $(\nu/4t)$.

4 Further Examples

Most of the preceeding examples have involved flows where the convection term vanished. That is not the case with the following examples. Some lie in R^2 and have no non-slip surface boundaries. They are useful to code developers for testing their codes against analytical solutions.

Flow in a Corner Due to a Point Source. Consider the steady two-dimensional laminar flow between two semi-infinite planes intersecting at an angle 2α, with a point source or sink at the intersection. This is a representation of the local flow in a converging or diverging channel. This problem was considered independently by Jeffery[5] and Hamel[2].

We introduce plane polar coordinates, with the origin at the intersection. We assume the flow is unidirectional in the radial direction, so $u_\theta = 0$. As the flow must conserve mass, we assume a radial velocity of the form,

$$u_r = Ar^{-1}F(\eta), \tag{51}$$

where $\eta = \theta/\alpha$. Assume F is dimensionless, so A is related to the flow rate. The no-slip boundary condition at the planes requires $F(\pm 1) = 0$. We assume F is normalized as $F(0) = 1$, and we have $F'(0) = 0$ by symmetry. Computing the convection and diffusion terms,

$$\begin{aligned} -\mathbf{u}\cdot\nabla\mathbf{u} &= \begin{bmatrix} A^2r^{-3}F^2 \\ 0 \end{bmatrix}, \\ \nu\nabla^2\mathbf{u} &= \begin{bmatrix} \nu A\alpha^{-2}r^{-3}F'' + 4\nu Ar^{-3}F \\ 0 \end{bmatrix} + \begin{bmatrix} -4\nu Ar^{-3}F \\ 2\nu A\alpha^{-1}r^{-3}F' \end{bmatrix}. \end{aligned} \tag{52}$$

A term $4\nu Ar^{-3}F$ has been added and subtracted to the first component of the diffusion term, so that the second part is irrotational (a term $\nu A\alpha^{-2}r^{-3}a$ could be added (and subtracted) for an additional pressure-driving force). Then from (5),

$$\begin{aligned} \pi^{\mathrm{S}}(-\mathbf{u}\cdot\nabla\mathbf{u} + \nu\nabla^2\mathbf{u}) &= \nu A\alpha^{-2}r^{-3}\begin{bmatrix} F'' + 4\alpha^2F + A\alpha^2\nu^{-1}F^2 \\ 0 \end{bmatrix} = \mathbf{0}\,, \\ \pi^{\mathrm{I}}(-\mathbf{u}\cdot\nabla\mathbf{u} + \nu\nabla^2\mathbf{u}) &= 2\nu A\begin{bmatrix} -2r^{-3}F \\ \alpha^{-1}r^{-3}F' \end{bmatrix} = 2\nu A\nabla(r^{-2}F) = \nabla p\,, \end{aligned} \tag{53}$$

The pressure equation is integrated to give (within a constant),

$$p(r,\theta) = 2\nu Ar^{-2}F(\cos\theta). \tag{54}$$

The velocity equation is made solenoidal by requiring the residual to vanish, giving,

$$F'' + 4\alpha^2F + \alpha ReF^2 = 0, \tag{55}$$

where we have introduced the Reynolds number $Re = \alpha A/\nu$. This can be recognized as a special case of the Jeffery-Hamel equation. When $Re = 0$, this can be integrated. Imposing boundary conditions gives,

$$u_r = \frac{A}{r}\,\frac{\cos 2\theta - \cos 2\alpha}{1 - \cos 2\alpha}. \tag{56}$$

For the general case, multiplying by F gives,

$$\tfrac{1}{2}(F'^2)' + 2\alpha^2(F^2)' + \tfrac{1}{3}\alpha Re(F^3)' = 0. \tag{57}$$

Integrating and introducing an integration constant c^2 and solving for F' gives,

$$F' = \pm(1-F)^{1/2}[\tfrac{2}{3}\alpha ReF(1+F) + 4\alpha^2F + c^2]^{1/2}, \tag{58}$$

where c is related to the shear stress at the walls. The final integration involves elliptic functions. The flow exhibits a variety of features at high Reynolds number, including a reversal of flow at the walls. References to this can be found in [8].

Kovasznay Flow. In this example, the problem domain is R^2, with the boundary condition $\mathbf{u} = 0$ at $(0, y_i)$, $y_i = 0, \pm 1, \pm 2, \ldots$. The stream function for the tentative solution is $\psi = y - (2\pi)^{-1}e^{\lambda x}\sin(2\pi y)$, with velocity components,

$$u = 1 - e^{\lambda x}\cos(2\pi y), \qquad v = \lambda/(2\pi)e^{\lambda x}\sin(2\pi y). \tag{59}$$

This solution, first proposed by L.I.G. Kovasznay [6] and known as Kovasnay flow, is identified with steady incompressible laminar flow behind a grid. A simple calculation gives,

$$\begin{aligned}\mathbf{u}\cdot\nabla\mathbf{u} &= \begin{bmatrix} -\lambda e^{\lambda x}\cos(2\pi y) \\ \lambda^2/(2\pi)e^{\lambda x}\sin(2\pi y)\end{bmatrix} + \begin{bmatrix} \lambda e^{2\lambda x} \\ 0\end{bmatrix},\\ \nu\nabla^2\mathbf{u} &= \begin{bmatrix} (4\pi^2-\lambda^2)e^{\lambda x}\cos(2\pi y) \\ -\lambda/(2\pi)(4\pi^2-\lambda^2)e^{\lambda x}\sin(2\pi y)\end{bmatrix}.\end{aligned} \tag{60}$$

The first term of the convection is solenoidal and the second irrotational, while the diffusion term is purely solenoidal. Applying (5),

$$\begin{aligned}\pi^{\mathrm{S}}(-\mathbf{u}\cdot\nabla\mathbf{u}+\nu\nabla^2\mathbf{u}) &= (\lambda-\nu(4\pi^2-\lambda^2))\begin{bmatrix} e^{\lambda x}\cos(2\pi y) \\ -\lambda/(2\pi)e^{\lambda x}\sin(2\pi y)\end{bmatrix} = 0\,,\\ \pi^{\mathrm{I}}(-\mathbf{u}\cdot\nabla\mathbf{u}+\nu\nabla^2\mathbf{u}) &= \begin{bmatrix} -\lambda e^{2\lambda x} \\ 0\end{bmatrix} = \nabla p\,,\end{aligned} \tag{61}$$

The requirement that the residual of the momentum term vanish leads to the condition $\lambda = Re/2 \mp \sqrt{Re^2/4+4\pi^2}$, where the solution with the upper (minus) sign is bounded (and vanishes) at $x = +\infty$, while with the lower (minus) sign it is bounded at $x=-\infty$. In this expression we have made the replacement $Re = 1/\nu$.

The pressure gradient derives from the irrotational residual of the convection term. With the boundary condition $p(0,y)=0$ the pressure is given by $p=\frac{1}{2}(1-e^{2\lambda x})$.

Taylor Vortex. This example involves a family of stream functions which are eigenfunctions of the Laplacian, $\nabla^2\psi_{mn} = -\lambda_{mn}\psi_{mn}$, with time dependence entering in a product form $\psi_{mn}(x,y)f_{mn}(t)$. They can be restricted to the finite domain by requiring that ψ_{mn} be periodic. This example was originated by G. I. Taylor [9] and extended by O. Walsh [10].

Take $\psi = -\pi^{-1}\cos\pi x\cos\pi y\exp(-\beta t)$. Then,

$$\mathbf{u} = e^{-\beta t}\begin{bmatrix} \cos\pi x\,\sin\pi y \\ -\sin\pi x\,\cos\pi y\end{bmatrix}. \tag{62}$$

By direct calculation,

$$\begin{aligned}\nu\nabla^2\mathbf{u} &= -2\pi^2\nu e^{-\beta t}\begin{bmatrix} \cos\pi x\,\sin\pi y \\ -\sin\pi x\,\cos\pi y\end{bmatrix} = -2\pi^2\nu\mathbf{u},\\ \mathbf{u}\cdot\nabla\mathbf{u} &= -\pi e^{-2\beta t}\begin{bmatrix} \cos\pi x\,\sin\pi x \\ \cos\pi y\,\sin\pi y\end{bmatrix},\\ \frac{\partial\mathbf{u}}{\partial t} &= -\beta\mathbf{u}\,.\end{aligned} \tag{63}$$

It is further seen by simple computation that the diffusion term is solenoidal and the convection term is irrotational. Substituting into the complementary velocity-pressure equations,

$$\begin{aligned}\beta &= 2\pi^2\nu,\\ \nabla p &= \pi e^{-2\beta t}\begin{bmatrix} \cos\pi x\,\sin\pi x \\ \cos\pi y\,\sin\pi y\end{bmatrix}\\ &= \tfrac{1}{2}e^{-2\beta t}\nabla(\sin^2\pi x+\sin^2\pi y).\end{aligned} \tag{64}$$

With $\beta = 2\pi^2\nu$ and choosing the pressure constant to be zero, it follows that

$$\begin{aligned}\mathbf{u} &= e^{-2\pi^2 t}\begin{bmatrix} \cos\pi x\,\sin\pi y \\ -\sin\pi x\,\cos\pi y\end{bmatrix},\\ p &= \tfrac{1}{2}e^{-4\pi^2\nu t}(\sin^2\pi x+\sin^2\pi y).\end{aligned} \tag{65}$$

See Gresho [1] and references therein for examples of studies of spacial accuracy of several finite elements and temporal accuracy and efficiency of several time integration methods.

5 Summary

While not exhaustive, the selection of problems presented here is sufficiently comprehensive to illustrate that the complementary pair of orthogonal velocity and pressure equations can be used in an introduction to fluid dynamics. The "projection by inspection" avoids the mathematical complexity of the explicit integro-differential equations as well as the abstractions of the weak formulation.

Apart from rederiving/verifying standard results, the method can clearly delineate the features of the incompressible flow which evoke the complementary pressure.

While this approach is a departure from practices of the past 150 years, it reflects the correct physics and mathematics of this idealized flow.

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References

[1] P. M. Gresho and R. L. Sani, *Incompressible Flow and the Finite Element Method*, John Wiley & Sons, Ltd., New York, 1998.

[2] G. Hamel, "Spiralförmige Bewegungen zäher Flüssigkeiten," *Jaresb. Deutschen Math. Vereinigung*, **25** (1916) 34-60.

[3] J. T. Holdeman, "I. Some Lagrange interpolation functions for solenoidal and irrotational vector fields," (*Submitted to Mathematics of Computation*, January 2003, *this can be viewed at http://j.t.holdeman.home.att.net/research.htm*).

[4] J. T. Holdeman, "II. Some Hermite interpolation functions for solenoidal and irrotational vector fields," (*Submitted Mathematics of Computation*, January 2003, *this can be viewed at http://j.t.holdeman.home.att.net/research.htm*).

[5] G. B. Jeffery, "The two-dimensional steady motion of a viscous fluid," *Phil. Mag.* **29** (1915) 455-465.

[6] L. I. G. Kovasznay, "Laminar flow behind a two-dimensional grid," *Proceedings of the Cambridge Philosophical Society*, **44** (1948) 58-62.

[7] J. Leray, "Etude de diverses équations intégrales nonlinéaires et de quelques problèmes que pose l'hydrodynamique," *J. Math. Pures Appl.* **12** (1933) 1–82.

[8] C. Pozrikidis, *Introduction to Theoretical and Computational Fluid Dynamics*, Oxford University Press, New York, 1997.

[9] G.I. Taylor, "On the decay of vortices in a viscous fluid," *Phil Mag. S. 6.*, **46** (1929) 271.

[10] O. Walsh, "Eddy solutions of the Navier-Stokes equations," *The Navier-Stokes Equations II – Theory and Numerical Methods: Conference Procedings*, J.G. Heywood, K. Masuda, R. Rautmann and V.A. Solonnikov, eds. Springer Verlag, Berlin, 1991.

[11] F. M. White, *Viscous Fluid Flow*, 2nd ed., McGraw-Hill, New York, 1991.