

# Governing Equation for Incompressible Flow; Revisiting the Navier-Stokes Equation

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It has been recognized by mathematicians for almost seventy years that the incompressible Navier-Stokes equation can be orthogonally decomposed into an equivalent pressureless governing equation for the fluid motion and an equation for the pressure as a functional of the velocity field. It would be troubling indeed if this decomposition did not have some “physical” basis. The “physical” basis is to be found from a reexamination of the derivation of the governing equation from first principles. Correction of an oversight leads to a pair of complementary equations more fundamental than the composite Navier-Stokes equation. It is then shown how fluid mechanics and computational fluid dynamics can be reconciled with this (new?) context.

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For some 150 years the Navier-Stokes equation (NSE) has been regarded as *the fundamental* governing equation for incompressible fluid flow, a view that is challenged in this work. Indeed, the aspect being challenged originated 100 years earlier, with the equation conceived by L. Euler for inviscid flow (the Euler equation). These original formulations applied to incompressible flow, as it was only *after* Navier, with the work of S. D. Poisson and later Stokes, that the Navier-Stokes equation was generalized to compressible flow[1].

Many compressible flows exhibit nearly constant density. This is true of most compressible flows with velocities below the Mach number  $M = 0.3$ . The limiting case of incompressible flow is associated with a hypothetical idealized incompressible fluid, which is assumed to exhibit a density independent of pressure. The derivation of the governing equations for incompressible flow is based on this model fluid.

The Navier-Stokes equation for incompressible laminar isothermal flow of a Newtonian fluid is stated in vector notation by

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p + \nu \Delta \mathbf{u} + \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \quad (1)$$

where  $\mathbf{u}$  is the fluid velocity,  $p = p'/\rho$  is the reduced pressure,  $\rho$  is the (constant) density,  $\nu$  the kinematic viscosity, and  $\mathbf{f}$  represents any body forces. The second (continuity) equation, expressing mass conservation, is implied in the rest of this paper. The fluid is assumed to fill a domain  $\Omega$  with boundary  $\partial\Omega = \Gamma = \Gamma_D + \Gamma_N$ . The flow is specified by the no-slip conditions  $\mathbf{u} = \mathbf{g}$  on Dirichlet boundaries  $\Gamma_D$  where  $\mathbf{g}$  is the velocity on the (perhaps moving) boundary, and  $\oint_{\Gamma} \mathbf{u} \cdot \mathbf{n} d\Omega = 0$  if  $\Gamma_N = 0$ . The fluid is assumed to satisfy the initial condition  $\mathbf{u}(\mathbf{x}, t = 0) = \mathbf{u}_0(\mathbf{x})$ , where  $\mathbf{u}_0$  is solenoidal. These

formalities aside, the problem with equation (1) will now be shown.

Equation (1) would seem to be a *dynamic* equation with the pressure as a driving force. That this is not so was first demonstrated by the mathematician J. Leray some 70 years ago[8][9].

Leray showed that the Navier-Stokes equation is a composite equation, a fact that has been used to advantage by mathematicians in proving results about the existence and uniqueness of solutions. His results will be demonstrated, in the strong form and then in the weak form after some preliminaries.

Let  $\pi^S$  and  $\pi^I$  be solenoidal and irrotational projection operators. As projection operators,  $\pi^S \pi^S = \pi^S$  and  $\pi^I \pi^I = \pi^I$ . The Helmholtz decomposition theorem states that any vector field satisfying appropriate boundary conditions (B.C.) can be *orthogonally* decomposed into a solenoidal part (with zero divergence) and an irrotational part (with zero curl). Then  $\pi^S + \pi^I = 1$  and  $\pi^S \pi^I = \pi^I \pi^S = 0$ . This means that  $\pi^S \nabla \phi = 0$  for any scalar field  $\phi$  and  $\pi^I \nabla \times \psi = 0$  for any vector field  $\psi$ . Applying these operators to the NSE (1) gives, after some simplification,

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{u} &= \pi^S (-\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u}) + \mathbf{f}^S, \\ \nabla p &= \pi^I (-\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u}) + \mathbf{f}^I, \end{aligned} \quad (2)$$

where the body force  $\mathbf{f}$  has been decomposed into non-conservative (solenoidal) and conservative (irrotational) parts  $\mathbf{f}^S$  and  $\mathbf{f}^I$  respectively. The first equation describes the time development of the fluid flow without reference to the pressure gradient or conservative forces, and the second equation describes the pressure field as a function(al) of the velocity field, and not vice versa.

Development of the weak form of (2) proceeds without reference to projection operators. However, it uses a result from the Helmholtz decomposition theorem that any solenoidal field  $\mathbf{u}$  and any irrotational field  $\mathbf{w}$  are orthog-

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onal over the problem domain  $\Omega$  (subject to B.C.).

$$(\mathbf{w}, \mathbf{u}) = (\mathbf{u}, \mathbf{w}) \equiv \int_{\Omega} \mathbf{w} \cdot \mathbf{u} d\Omega = 0. \quad (3)$$

This result can be demonstrated by writing  $\mathbf{u}$  as the curl of a vector potential and  $\mathbf{w}$  as the gradient of the scalar potential, integrating by parts, and using the vector identities  $\nabla \cdot \nabla \times \mathbf{u} = 0$  for any vector  $\mathbf{u}$  or  $\nabla \times \nabla \phi = 0$  for any scalar  $\phi$ , subject to a B.C. which make the surface integral vanish.

Now if (1) is true, then it must be true when multiplied by any sufficiently-continuous and integrable function  $\mathbf{w}$  and integrated over the problem domain  $\Omega$ ,

$$\int_{\Omega} \mathbf{w} \cdot \left( \frac{\partial}{\partial t} \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \Delta \mathbf{u} - \mathbf{f} \right) d\Omega = 0. \quad (4)$$

Conversely if (4) is true for all functions in a sufficiently-complete set or space of (test or weight) functions and  $\mathbf{u}$  is sufficiently continuous, then (1) is true. In particular, taking  $\mathbf{v}$  to be any member of a space of *solenoidal* functions and  $\mathbf{w}$  any member of a space of *irrotational* functions and using orthogonality yields,

$$\begin{aligned} (\mathbf{v}, \frac{\partial}{\partial t} \mathbf{u}) &= -(\mathbf{v}, \mathbf{u} \cdot \nabla \mathbf{u}) - \nu (\nabla \mathbf{v}, \nabla \mathbf{u}) + (\mathbf{v}, \mathbf{f}^S), \\ (\mathbf{w}, \nabla p) &= -(\mathbf{w}, \mathbf{u} \cdot \nabla \mathbf{u}) - \nu (\nabla \mathbf{w}, \nabla \mathbf{u}) + (\mathbf{w}, \mathbf{f}^I). \end{aligned} \quad (5)$$

These are an equivalent formulation [2][13] of the strong forms (2). The discussion above can be made more mathematically-precise at additional cost in notational complexity with little gain in insight.

Mathematicians seem to have regarded this decomposition as a convenient “mathematical trick.” If the decomposition were not reflected in the physics of the problem, this would be a serious conflict indeed. But the pressure gradient has been long been accepted as a term of the governing equation and its derivation is part of any elementary physics course. Surely these derivations can’t be wrong, but perhaps the conviction that pressure must drive the flow has resulted in just that.

Before proceeding with a (false) derivation of the offending pressure gradient term, it must reiterated that the model of an incompressible fluid being used is an idealization which does not exist in nature, and certainly not in common experience. Consequently, caution must be used in applying intuition regarding how such a material would behave.

While mathematicians since J. Leray [2][10][11][13] have understood the possibility and utility of orthogonally decomposing the NSE, none seem to have asserted the decomposition as *fundamental* and *necessary* from a physical point of view. Now consider the classical derivation of the equation of motion of an incompressible fluid from the physical perspective using the balance of momentum of the fluid in a small volume  $\Delta V = \Delta x \Delta y \Delta z$ . By Newton’s second law, the change in momentum per unit time of the fluid in the volume involves flow in and out of the volume and the sum of the forces acting on the

enclosed fluid. It is argued that one force results from a pressure difference across opposite faces of the volume element. The difference in force on the two faces normal to the  $x$ -axis, per unit volume, is claimed to be

$$-\frac{(p_2 - p_1) \Delta y \Delta z}{\Delta x \Delta y \Delta z} = -\frac{\Delta p}{\Delta x} \sim -(\nabla p)_x. \quad (6)$$

However, since pressure disturbances in an incompressible medium propagate at an infinite velocity, a *dynamic* (non-hydrostatic) pressure differential across the volume would be *instantly* equilibrated. As a *dynamic* pressure difference cannot exist for finite time, there can be no pressure gradient term in the equation of motion for an incompressible fluid. Likewise, only the nonconservative body forces can affect the change in momentum [4][5]. Thus it is postulated that the fundamental equation of motion of an incompressible fluid is given formally by,

$$\frac{\partial}{\partial t} \mathbf{u} = \pi^S (-\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u}) + \mathbf{f}^S. \quad (7)$$

The projection operator is necessary because the velocity on the left is solenoidal, while the operand of the operator on the right is not.

The *effective* or *complementary* pressure is defined as a function of the flow and any conservative body forces by (an algebraic equation in time),

$$\nabla p = \pi^I (-\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u}) + \mathbf{f}^I. \quad (8)$$

This equation is the *complement* of (7) in the sense that summing the two eliminates the projection operators and results in the NSE. The apparent simplicity of the NSE obscures its composite nature and the fact that it is not a simple differential equation in time, but a differential algebraic equation (DAE), which is well known to pose difficulties when integrating it in time.

The equation (7) is to be regarded as the *fundamental equation of motion* of an incompressible fluid. It is more fundamental than the NSE because the NSE is a composite equation. When truncating the physical domain  $R^3$  to the computational domain  $\Omega$ , the B.C. are a consequence of (7) rather than the decomposition depending on the B.C. It is clear that (7) is a *kinematic* equation (in the sense of classical mechanics) with the incompressibility condition serving the role of a conservation law or invariant.

The projection operators can be written formally as[3],

$$\pi^I = \nabla \Delta^{-1} \nabla \cdot, \quad \pi^S = I - \pi^I, \quad (9)$$

where  $\Delta^{-1}$  is the inverse of the Laplacian operator. This inverse can be expressed in terms of the Green’s function for the Laplacian, which leads to the explicit integro-differential equations. These forms satisfy all of the properties of the projection operators, as can be shown algebraically.

For simple boundaries, explicit forms of the operators may be found. For instance in infinite three-dimensional

space  $R^3$ , if  $\mathbf{F} = 0$  at infinity, then [12],

$$\pi^I \mathbf{F}(x, y, z) = -\nabla \int_{R^3} \frac{1}{4\pi r} \nabla' \cdot \mathbf{F}(x', y', z') d\Omega', \quad (10)$$

where  $r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$ .

These forms are difficult to work with in computations where the weak form of the complementary equations may be preferred.

A question arises whether an introductory fluid mechanics course can be taught at the undergraduate level using the complementary  $u - p$  equations. In fact, all elementary examples, all known analytical solutions to the NSE, can be developed without more-advanced mathematics using the method of ‘‘projection by inspection.’’ [6] With this method, the convection and diffusion terms are computed for a trial solution and separated into solenoidal and irrotational parts by inspection. These parts are substituted into (2) and the resulting equations solved. In practice, one needs to work through a number of examples to develop a ‘‘bag of tricks’’ for solving these problems.

Most analytical solutions have some simplifying feature such as the unidirectional flow, or allow some form of linearization of the convection term. The example considered here is a case of Beltrami flow, where the vorticity is parallel to the flow.

In this example, the problem domain is  $R^2$ , with the B.C.  $\mathbf{u} = 0$  at  $(0, y_i)$ ,  $y_i = 0, \pm 1, \pm 2, \dots$ . The stream function for the tentative solution is  $\psi = y - (2\pi)^{-1} e^{\lambda x} \sin(2\pi y)$ , with velocity components,

$$u = 1 - e^{\lambda x} \cos(2\pi y), \quad v = \lambda/(2\pi) e^{\lambda x} \sin(2\pi y). \quad (11)$$

This solution, first proposed by L.I.G. Kovasznay [7], is identified with steady incompressible laminar flow behind a grid. By simple calculation,

$$\begin{aligned} \mathbf{u} \cdot \nabla \mathbf{u} &= \begin{bmatrix} -\lambda e^{\lambda x} \cos(2\pi y) \\ \lambda^2/(2\pi) e^{\lambda x} \sin(2\pi y) \end{bmatrix} + \begin{bmatrix} \lambda e^{2\lambda x} \\ 0 \end{bmatrix}, \\ \nabla^2 \mathbf{u} &= \begin{bmatrix} (4\pi^2 - \lambda^2) e^{\lambda x} \cos(2\pi y) \\ -\lambda/(2\pi) (4\pi^2 - \lambda^2) e^{\lambda x} \sin(2\pi y) \end{bmatrix}. \end{aligned} \quad (12)$$

The first term of the convection is solenoidal and the second irrotational, while the diffusion term is purely solenoidal. Applying (2),

$$\begin{aligned} \pi^S(-\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u}) &= (\lambda - \nu(4\pi^2 - \lambda^2)) \\ &\times \begin{bmatrix} e^{\lambda x} \cos(2\pi y) \\ -\lambda/(2\pi) e^{\lambda x} \sin(2\pi y) \end{bmatrix} = 0, \quad (13) \\ \pi^I(-\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u}) &= \begin{bmatrix} -\lambda e^{2\lambda x} \\ 0 \end{bmatrix} = \nabla p, \end{aligned}$$

The requirement that the residual of the momentum term vanish leads to the condition  $\lambda = Re/2 \mp \sqrt{Re^2/4 + 4\pi^2}$ , where the solution with the upper (minus) sign is bounded (and vanishes) at  $x = +\infty$ , and with the lower (plus) sign it is bounded at  $x = -\infty$ . The replacement  $Re = 1/\nu$  has been made in this expression.

The pressure gradient in this case derives from the irrotational residual of the convection term. With the B.C.  $p(0, y) = 0$ , the pressure is given by  $p = \frac{1}{2}(1 - e^{2\lambda x})$ . Other examples can be found in [6].

Weak projections will be discussed here in the context of the finite element method (FEM), but the results apply to other numerical methods as well. In this case, solutions of fundamental governing equation (2) are often found using mixed methods. Using  $\pi^S = 1 - \pi^I$ , the first equation in (2) is written as,

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{u} &= (1 - \pi^I)(-\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u}) + \mathbf{f}^S \\ &= -\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u} + \mathbf{f}^S - \pi^I(-\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u}) \\ &= -\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u} + \mathbf{f}^S - \nabla \phi, \end{aligned} \quad (14)$$

where,

$$\nabla \phi = \pi^I(-\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u}). \quad (15)$$

The remaining projection operator  $\pi^I$  can be eliminated by taking the divergence of  $\nabla \phi$ ,

$$\nabla^2 \phi = \nabla \cdot (-\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u}). \quad (16)$$

The pair of equations to be solved is,

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{u} &= -\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u} + \mathbf{f}^S - \nabla \phi, \\ \nabla^2 \phi &= \nabla \cdot (-\mathbf{u} \cdot \nabla \mathbf{u} + \nu \nabla^2 \mathbf{u}). \end{aligned} \quad (17)$$

It looks like the pressure has been reintroduced calling it  $\phi$ , but not everything that looks like the pressure is a pressure. Boundary conditions on  $\phi$  must preserve the B.C. on  $\mathbf{u}$ .

The weak form is found by multiplying (14) and (15) by appropriate weight functions and integrating over the problem domain. Instead of using the divergence to eliminate the projection operator as was done in (16), the orthogonality of the gradients of a scalar function space of test (weight) functions is used. For discrete computations, sets of finite element bases are introduced on a discretization  $\Omega^h$  of the problem domain  $\Omega$ . The weak discrete form of the governing equation and the projection equation are given by,

$$\begin{aligned} (\mathbf{v}^h, \dot{\mathbf{u}}^h) &= -(\mathbf{v}^h, \mathbf{u}^h \cdot \nabla \mathbf{u}^h) - \nu(\nabla \mathbf{v}^h, \nabla \mathbf{u}^h) \\ &\quad + (\mathbf{v}^h, \mathbf{f}^S) - (\mathbf{v}^h, \nabla \phi^h), \quad (18) \\ (\nabla g^h, \nabla \phi^h) &= -(\nabla g^h, \mathbf{u}^h \cdot \nabla \mathbf{u}^h) + \nu(\nabla g^h, \nabla^2 \mathbf{u}^h), \end{aligned}$$

where  $\phi^h, g^h, \mathbf{u}^h$  and  $\mathbf{v}^h$  lie in the spaces spanned by the chosen discrete bases. Care must be exercised in the choice of basis for  $\phi^h$  to assure the existence and uniqueness of solutions in the stationary case. The pressure is recovered as needed from,

$$\begin{aligned} (\nabla q^h, \nabla p^h) &= -(\nabla q^h, \mathbf{u}^h \cdot \nabla \mathbf{u}^h) \\ &\quad + \nu(\nabla q^h, \nabla^2 \mathbf{u}^h) + (\nabla q^h, \mathbf{f}^I), \end{aligned} \quad (19)$$

with  $q^h$  and  $p^h$  in the space spanned by the discrete pressure basis and  $\mathbf{u}^h$  (weakly) solenoidal from (18).

It remains to resolve the apparent dilemma of the pressureless computation of “pressure-driven” flows. A typical example is steady flow in a pipe or channel. In the standard view, the pressure gradient is the data controlling the flow. That data should be replaced by  $\int_{\Gamma_{N_i}} \mathbf{u} \cdot \mathbf{n} d\Omega$  over the inflow (or outflow) boundary  $\Gamma_{N_i}$ , which in two dimensions is  $\Delta\Psi$ , the difference in stream function across the channel.

For fully-developed channel flow, the pressure gradient is proportional to the flow. In two dimensions the relation can be written  $|\nabla p| = c_p \Delta\Psi$ , where  $c_p$  is known as the Poiseuille constant [1]. A similar relation exists in three dimensions, where the flow is given by the line integral of the tangential component of the vector potential around

the flow. The inappropriateness of pressure in traction B.C. is left as an exercise for the reader.

The NSE cannot be justified from its classical derivation and no mixed methods derived from it are entirely satisfactory. Recently, strongly solenoidal Hermite bases for the FEM have been found [5] which, by contrast, make incompressible CFD exceedingly simple. With these bases the NSE pressure and continuity matrices are identically zero. This singular situation is resolved by starting from (7). Not only are there fundamental arguments for starting from (7), but it is necessary when using basis functions most appropriate to the incompressible flow problem.

The problem with the classical derivation of the incompressible NSE is irrefutable, and appropriate adjustments should be made in the physics curriculum.

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