Incompressible Flow – Dynamic or Kinematic?

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Abstract

The Navier–Stokes equations for an incompressible fluid are orthogonally decomposed into an equation for fluid flow that does not contain pressure and an equation for pressure as a function of flow. This shows that incompressible flow is a kinematic problem, with the incompressibility serving as an underlying conservation law.

Key words: incompressible flow, kinematic flow, Navier–Stokes, orthogonal decomposition, Burgers equation
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The initial formulation of the governing equations for fluid flow appeared some 250 years ago. At that time, fluid friction was recognized, but its mathematical formulation was not understood. But by the mid-nineteenth century, the classical theory of nonturbulent Newtonian flow was essentially complete.

The basic equations for the flow of a Newtonian fluid, known as the Navier–Stokes equations, are given in terms of the density ρ, velocity $\mathbf{u}$, coefficient of viscosity μ, and body force $\mathbf{f}$, by

$$\begin{aligned} \tfrac{\partial}{\partial t}\rho &= -\nabla \cdot \rho\mathbf{u}\,, \\ \tfrac{\partial}{\partial t}\rho\mathbf{u} + \mathbf{u}\cdot\nabla\rho\mathbf{u} + \nabla p - \mu\Delta\mathbf{u} &= \mathbf{f}\,. \end{aligned} \qquad (1)$$

The first equation expresses conservation of mass, and the second, conservation of momentum. These equations may be supplemented with an energy conservation equation.

A material-dependent equation of state is required for closure with compressible flow, but closure may also be obtained by assuming incompressibility. Although incompressibility is an idealization, compressible fluids can exhibit incompressible *flow* at velocities that are low compared with the speed of sound in the fluid (i.e., at small Mach numbers). The assumption of incompressibility eliminates the need for an equation of state and reduces the mass-conservation equation to the constraint $\nabla\cdot\mathbf{u} = 0$, requiring the flow field to be divergence-free.

Assume isothermal flow with specified initial and boundary conditions. After dividing by ρ, the incompressible form of the Navier–Stokes (INS) equations over a problem domain Ω is given by

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} \;+\; \nabla p - \nu\Delta\mathbf{u} &= \mathbf{f} \text{ in } \Omega\,, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega\,, \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \partial\Omega\,, \\ \mathbf{u}(\mathbf{x},0) &= \mathbf{u}_0(\mathbf{x}) \quad \text{at } t=0\,, \end{aligned} \qquad (2)$$

where the reduced pressure p/ρ will simply be denoted by p and referred to as "the pressure," and $\nu = \mu/\rho$ is known as the kinematic viscosity.

If the second (nonlinear) advection term is absent (i.e., at low-enough velocities), the resulting equations are known as the Stokes equations. In the absence of the pressure gradient, the equations are an incompressible form of the Burgers equation; in the absence of the viscous (diffusion) term, they are called the Euler equations.

The arguments to follow examine the basis of

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incompressible flow, and are written for the non-specialist in computational fluid dynamics. Mathematical rigor in the discussion may be relaxed for clarity, but the results can be rigorously verified.

Using the methods of mathematical functional analysis, one defines a vector space, $\mathbf{V}$, of sufficiently continuous functions, each satisfying the boundary conditions, and a divergence-free (solenoidal) subspace, $\mathbf{S}$. As the divergence-free constraint is implied in the definition of the space $\mathbf{S}$, the flow is described by the single differential-algebraic equation,

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = \mathbf{f} \text{ in } \Omega\,, \qquad \mathbf{u}\in\mathbf{S}\,, \tag{3}$$

where the specification of the initial condition has been omitted. The spaces $\mathbf{V}$ and $\mathbf{S}$ are given the usual inner product $(\mathbf{v},\mathbf{u})$ and norm $(\mathbf{u},\mathbf{u})^{1/2}$ over Ω, and so are Hilbert spaces.

The inner product of equation (3) with every function (or every basis function) $\mathbf{v}$ in $\mathbf{V}$ is formed, where the inner products involve integrals over the problem domain Ω. The result may be written as

$$(\mathbf{v},\dot{\mathbf{u}}) + (\mathbf{v},\mathbf{u}\cdot\nabla\mathbf{u}) + (\mathbf{v},\nabla p) - \nu(\mathbf{v},\Delta\mathbf{u}) = (\mathbf{v},\mathbf{f}) \quad \text{in } \Omega\,, \quad \mathbf{u}\in\mathbf{S}\,, \quad \forall\mathbf{v}\in\mathbf{V}\,. \tag{4}$$

The pressure-gradient term and the diffusion term are often integrated by parts to reduce continuity requirements. This inner product form is known as the variational or weak formulation of the Navier–Stokes equations and the solutions as weak or generalized solutions.

Any solution of (3) satisfies (4). If (4) is satisfied for every $\mathbf{v}$ in $\mathbf{V}$, and $\mathbf{u}$ is sufficiently smooth, it can be shown that a solution of (4) is also a solution of (3). Thus the differential equation (3) and the variational equation (4) are equivalent descriptions of incompressible flow.

About seventy years ago, a proof of the existence and uniqueness of solutions to the incompressible Navier–Stokes equation was developed by the mathematician J. Leray. In a series of papers [1–3] published in 1933–34, Leray showed that the solution of a variational equation which contains no pressure gradient term results in exactly the same *flow* as given by the variational equation (4) and thus by equation (3). Later studies by mathematicians J. Lions[4–6], M. Fortin[7], R. Temam[8], V. Girault and P. A. Raviart[9] firmly established the remarkable pressure-free formulation for incompressible flow:

$$(\mathbf{v},\dot{\mathbf{u}}) + (\mathbf{v},\mathbf{u}\cdot\nabla\mathbf{u}) - \nu\,(\mathbf{v},\Delta\mathbf{u}) = (\mathbf{v},\mathbf{f}) \text{ in } \Omega, \quad \mathbf{u}\in\mathbf{S}\,, \ \forall\mathbf{v}\in\mathbf{S}\,. \tag{5}$$

Incompressible flow can be calculated using this equation without reference to the pressure. This formulation has been generally regarded as a mathematical "trick" to prove the existence of solutions. It will be shown that this little-appreciated result is fundamental.

To proceed further, it is necessary to digress for a moment to add some mathematical tools. There exists the well-known result that any vector field $\mathbf{V}$ can be orthogonally decomposed as $\mathbf{V} = \nabla\times\mathbf{A} + \nabla\phi$, where ϕ is a scalar field known as the scalar potential, and $\mathbf{A}$ is a divergence-free vector field known as the vector potential. In other words, any vector field may be decomposed into solenoidal and irrotational parts, where the solenoidal part is given by $\nabla\times\mathbf{A}$ and the irrotational part is given by $\nabla\phi$.

Explicit decomposition of an arbitrary vector field is not straightforward. However, one can formally define projection operators π^S and π^I with the properties that, for an arbitrary vector field $\mathbf{V}$, $\pi^S\mathbf{V}$ returns the solenoidal part of $\mathbf{V}$ and $\pi^I\mathbf{V}$ returns the irrotational part. It follows that $\pi^S\pi^S = \pi^S$ and $\pi^I\pi^I = \pi^I$, so the operators have eigenvalues of 0 and 1. The product $\pi^S\pi^I = 0$ expresses the orthogonality of the projections. Formally, the operators can be written as

$$\pi^I = \nabla\Delta^{-1}\nabla\cdot\,, \quad \pi^S = I - \pi^I,$$

or

$$\begin{aligned}\pi^S &= \mathbf{curl}\,(\mathbf{curl}\;\mathbf{curl})^{-1}\mathbf{curl}\\ &= -\mathbf{curl}\,\Delta^{-1}\mathbf{curl}\,, \quad \pi^I = I - \pi^S.\end{aligned} \tag{6}$$

The vector identity $\nabla\times\nabla\times\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \Delta\mathbf{u}$ has been used in the second line of the second operator. It is easily verified that these expressions satisfy the required conditions.

The inverse Laplacian operators are related to the Green's function for the Poisson equation. Explicit forms can be found for simple boundaries and boundary conditions. For instance, if $\mathbf{F} = 0$ at infinity (homogeneous Dirichlet boundary conditions), then in three-dimensional space[10],

$$\pi^S\mathbf{F}(x,y,z) = \nabla\times\int_{R^3}\frac{1}{4\pi r}\nabla'\times\mathbf{F}(x',y',z')\ \mathrm{d}\Omega',$$

$$\pi^I \mathbf{F}(x,y,z) = -\nabla \int_{R^3} \frac{1}{4\pi r} \nabla' \cdot \mathbf{F}(x',y',z')\ \mathrm{d}\Omega',$$

$$\text{where} \quad r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 . \quad (7)$$

The forms are similar in two dimensions with $1/4\pi r$ replaced by $\log(1/r^2)$. This completes the digression.

Returning to the INS equation, one seeks an orthogonal decomposition of (2). If $\mathbf{u}$ is solenoidal, so too are $\dot{\mathbf{u}}$ and $\Delta\mathbf{u}$. If the body force $\mathbf{f}$ is decomposed into a solenoidal (nonconservative) part $\mathbf{f}^S$ and an irrotational (conservative) part $\mathbf{f}^I$, then for solenoidal flow, (2) can be decomposed as

$$(\dot{\mathbf{u}} + \pi^S(\mathbf{u}\cdot\nabla\mathbf{u}) - \nu\Delta\mathbf{u} - \mathbf{f}^S) + (\nabla p + \pi^I(\mathbf{u}\cdot\nabla\mathbf{u}) - \mathbf{f}^I) = 0 . \quad (8)$$

As the two terms are orthogonal, they vanish separately. Then the velocity field is governed by

$$\dot{\mathbf{u}} + \pi^S(\mathbf{u}\cdot\nabla\mathbf{u}) - \nu\Delta\mathbf{u} = \mathbf{f}^S , \quad \mathbf{u} \in \mathbf{S} . \quad (9)$$

This equation can be recognized as the incompressible form of the Burgers equation, but it will also be called the *kinematic form of the Navier–Stokes equation*(KNS), for reasons to be discussed later.

The second term in equation (8) can be put in alternative forms

$$\nabla p = -\pi^I(\mathbf{u}\cdot\nabla\mathbf{u}) + \mathbf{f}^I , \\ = -\frac{1}{2}\nabla(u^2) + \mathbf{f}^I + \pi^I(\mathbf{u}\times\nabla\times\mathbf{u}) \quad (10)$$

using the vector identity $\mathbf{u}\cdot\nabla\mathbf{u} = \nabla u^2/2 - \mathbf{u}\times\nabla\times\mathbf{u}$. This is a differential form of the Bernoulli equation. The pressure gradient is a function of the advection term of the flow field given by (9) and is independent of the rate of change of momentum or acceleration term and of the diffusion term.

Substituting $\mathbf{f}^I = -\nabla\phi_f$ and using an explicit form for the projection operator, the pressure gradient can be integrated to give the Bernoulli equation for unsteady, viscous, incompressible flow:

$$p - p_r = -u^2/2 - \phi_f + \mathbf{\Delta}^{-1}\nabla\cdot(\mathbf{u}\times\nabla\times\mathbf{u}) , \quad (11)$$

where p_r is an integration constant or reference pressure. This result is established without reference to stream lines or tubes.

The orthogonal decomposition demonstrates that the Navier–Stokes equation is not a single dynamic equation for the velocity, where the pressure drives the flow, but rather a combination of two equations— the second giving the pressure as a derived variable that arises as a consequence of the flow.

These equations are somewhat awkward for analysis and computation because of the presence of the projection operators. This will be remedied below.

If the weight or test functions in the variational form (4) vary only over the space of solenoidal functions $\mathbf{S}$, then the inner product with the pressure gradient, the conservative parts of the body force, and irrotational part of the advection term vanish by orthogonality, leaving

$$(\mathbf{v}, \dot{\mathbf{u}}) + (\mathbf{v}, \mathbf{u}\cdot\nabla\mathbf{u}) - \nu(\mathbf{v}, \Delta\mathbf{u}) = (\mathbf{v}, \mathbf{f}^S) , \\ \mathbf{u} \in \mathbf{S} , \quad \forall \mathbf{v} \in \mathbf{S} . \quad (12)$$

This equation is the variational form of equation (9). Again, any solution of (9) is obviously a solution of (12), and it can be shown that any sufficiently smooth solution of (12) is a solution of (9). The two are equivalent descriptions of incompressible flow, and by previous arguments the flow is equivalent to the flow determined by (3) or (4). The advantage of (12) is that if solenoidal forms for the weight functions $\mathbf{v}$ are known, explicit forms of the projection operators are not needed.

Now consider a space $\mathbf{I}$ of irrotational functions, sufficiently continuous, and vanishing on the problem boundary. Then $\mathbf{V} = \mathbf{S} \oplus \mathbf{I}$, and any function in $\mathbf{I}$ is orthogonal to any function in the solenoidal space $\mathbf{S}$. If the weight functions $\mathbf{v}$ are taken from $\mathbf{I}$, the inner products $(\mathbf{v}, \dot{\mathbf{u}})$, $(\mathbf{v}, \Delta\mathbf{u})$, and $(\mathbf{v}, \mathbf{f}^S)$ vanish by orthogonality, resulting in an equation for the pressure gradient,

$$(\mathbf{v}, \nabla p) = -(\mathbf{v}, \mathbf{u}\cdot\nabla\mathbf{u}) + (\mathbf{v}, \mathbf{f}^I) , \\ \mathbf{u} \in \mathbf{S} , \quad \forall \mathbf{v} \in \mathbf{I} . \quad (13)$$

This equation is the variational form of (10).

Now assume (i) incompressible flow is governed by a dynamic equation with the pressure gradient an internal force, and (ii) Eq. (2) is that dynamic equation. An inconsistency with the classical theory will be demonstrated.

Consider the flow field governed by (2) at some instant of time and the rate of change of the momentum of a quantity of fluid in a small volume. Subtract from that rate, the rate of change due to transport of momentum by advective and diffusive means and any body forces. The resulting rate, if non-zero, would be due to a pressure gradient. But equation (10), which follows from (2), shows that the pressure gradient and the rate of change of momentum are independent; hence the assumption of a dynamic pressure gradient component governed by (2) is incorrect.

Conversely, assuming a dynamic governing equation and the correctness of the classical derivation of the momentum equation, one is led to (2). Thus an error must have been made in either the dynamic assumption or the derivation.

Actually, both are flawed. To influence the rate of change of momentum, a pressure gradient must be established which is unbalanced by other forces acting in the problem. However, pressure (sound) waves propagate at infinite speed in an incompressible medium, so any dynamic pressure difference is instantly equilibrated. Thus the conditions necessary for a dynamic description cannot be established, so a dynamic description cannot exist.

The constraint that the flow be divergence-free can be considered as a conservation law. In a sense, the fluid behaves as a single shapeless mass. Internal forces due to pressure gradients cancel by Newton's third law. This is the prescription for a kinematic problem. The following conclusions summarize the results.

INCOMPRESSIBLE FLOW

The condition $\nabla \cdot \mathbf{u} = 0$ plays the role of a conservation law, eliminating the need for a detailed description of the internal pressure forces. The assumption of incompressibility allows a much simpler *kinematic* description of the fluid in the idealized system. A consistent "Bernoulli" pressure can be found which is a function of the flow velocity and external conservative forces.

CONCLUSION 1. *The governing equation for incompressible fluid flow is the incompressible Burgers equation/kinematic Navier–Stokes equation [Eq. (9)], or its variational form [Eq. (12)].*

CONCLUSION 2. *The governing equation for the pressure consistent with a divergence-free flow field is the differential form of the Bernoulli equation [Eq. (10)], or its variational form [Eq. (13)].*

CONCLUSION 3. *In three dimensions with homogeneous boundary conditions on Ω, the flow and pressure are governed by*

$$\dot{\mathbf{u}} = \nu\Delta\mathbf{u} + \mathbf{f}^S - \nabla \times \int_\Omega \frac{1}{4\pi r}\nabla' \times (\mathbf{u} \times \nabla' \times \mathbf{u})\, \mathrm{d}\Omega',$$

$$p - p_r = -\phi_f + \int_\Omega \frac{1}{4\pi r}\nabla'\mathbf{u} : \nabla'\mathbf{u}\, \mathrm{d}\Omega',$$

$$r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2,$$

where $\mathbf{f}^I = -\nabla\phi_f$ and $(:)$ indicates the contraction of two tensors. In two dimensions the quantities are given by similar expressions with $1/4\pi r$ replaced by $\log(1/r^2)$.

The variational equations that have been displayed, or rather their discrete forms, are the foundation for computation of fluid flow by the Galerkin finite element method. The finite element method is perhaps the most versatile way to handle incompressible flow problems in diverse geometries. In this method, the flow field is expanded in terms of an interpolating basis set defined over a partition of the problem domain by a mesh.

Unfortunately, there have been no generally known (or rather recognized) discrete interpolation bases such that any interpolated flow field lies in $\mathbf{S}$. Instead, nonsolenoidal elements have been used in "mixed methods" to approximate the divergence constraint in a "weak" or average sense. Often projection methods are used which lead, at least implicitly, to discrete forms of the projection operators (6), but under rather restrictive conditions on the bases involved.

Recently, some local divergence-free basis functions of the Hermite type have been found[11]. Examples are now known for two and three dimensions in Cartesian and some curvilinear coordinate systems. These support truly simple finite element implementations for the solution of (12) and (13), with no need for projections.

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References

[1] J. Leray, Etude de diverses équations intégrales nonlinéaires et de quelques problèmes que pose l'hydrodynamique, *J. Math. Pures Appl.* **12** (1933) 1–82.

[2] J. Leray, Essai sur les mouvements d'un liquide visqueux que limitent des parois, *J. Math. Pures Appl.* **13** (1934) 331–418.

[3] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta. Math.* **63** (1934) 193–248.

[4] J. L. Lions, Sur l'existence des solutions des équations de Navier–Stokes, *C. R. Acad. Sci., Paris* **248** (1959) 2847–2849.

[5] J. L. Lions and G. Prodi, Un théorème d'existence et d'unicité dans les équations de Navier–Stokes en dimension 2, *C. R. Acad. Sci., Paris* **248** (1959) 3519–3521.

[6] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[7] M. Fortin, Calcul numérique des ècoulmentes des fluides de Bingham et des Newtoniens incompressibles par la méthode des éléments finis. Thèse, Univ. Paris (1972).

[8] R. Temam, *Navier–Stokes Equations, Theory and Numerical Analysis*, North-Holland Publishing Company, New York, 1977.

[9] V. Girault and P. A. Raviart, *Finite Element Approximation of the Navier–Stokes Equations*, Springer-Verlag, Berlin, 1979.

[10] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York, 1953.

[11] J. T. Holdeman (to be published).