Controllability and Local Accessibility—A Normal Form Approach

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***Abstract*—Given a system with an uncontrollable linearization at the origin, we study the controllability of the system at equilibria around the origin. If the uncontrollable mode is nonzero, we prove that the system always has other equilibria around the origin. We also prove that these equilibria are linearly controllable provided a coefficient in the normal form is nonzero. Thus, the system is qualitatively changed from being linearly uncontrollable to linearly controllable when the equilibrium point is moved from the origin to a different one. This is called a bifurcation of controllability. As an application of the bifurcation, systems with a positive uncontrollable mode can be stabilized at a nearby equilibrium point. In the last part of this paper, simple sufficient conditions are proved for local accessibility of systems with an uncontrollable mode. Necessary conditions of controllability and local accessibility are also proved for systems with a convergent normal form.**

***Index Terms*—Linearly controllable, nonlinear systems, normal forms, stabilizable.**

I. Introduction

IT IS WELL known that a system with an uncontrollable mode in the right-half plane is not stabilizable using smooth feedback. Stabilization by nonsmooth or time-dependent feedbacks was studied by many researchers. A large number of publications and elegant results can be found in the literature (see, for instance, [4]–[6], [9], [16], [17], and [19]–[22]).

In this paper, we study the controllability, stabilizability and local accessibility of systems with a single uncontrollable mode. The term *controllability* is used in this paper to represent the controllability of the linearized system. The viewpoint and approach adopted in this paper are fundamentally different from existing publications on nonsmooth or time-dependent feedback stabilization. Instead of focusing on the stability of a single equilibrium point, we study the existence and the controllability of all equilibria in a neighborhood of the point of interest. The theoretical approach in this paper is based on the normal form of nonlinear control systems, a relatively new theoretical tool that has been actively developed during the last ten years. Different from the results in [16], [17], [19], and [20], we do not restrict our attention on homogeneous or generalized triangular systems. The original system is not required to have any triangular structure. On the other hand, the stability achieved by the feedback in this paper is local, while the results from [16], [17], [19], and [20] are global. In addition, we do assume that the system has a single uncontrollable mode and a single input. Given the rapid development in the normal form theory of systems with multiple uncontrollable modes and multiple inputs, we hope that this restriction will be removed in the future.

Given a system with a nonzero uncontrollable mode and a linearly controllable part, we prove that there always exists an infinite number of equilibria around this point. More importantly, it is proved that the controllability of a system could change when the equilibrium point is varied. This qualitative change in controllability is called a *bifurcation* of the control system [12]. As a by-product of the bifurcation in controllability, a system with a positive uncontrollable mode at an equilibrium point may have infinitely many, arbitrarily close equilibrium points at which the system becomes linearly controllable. Thus, the system can be stabilized at a point sufficiently close to the desired equilibrium. In addition to the stabilization, we also proved a simple relationship between the normal form of a system and its local accessibility. Even if a system has uncontrollable mode, its local accessibility can be easily determined based on its normal form. Necessary conditions for controllability and local accessibility are also addressed.

This paper is organized as follows. Sections II and III focus on the bifurcation of controllability. Theorems are introduced in Section II. They are proved in Section III. In Section IV, examples are introduced for the problem of feedback stabilization. The feedback is designed based on the bifurcation of controllability. In Section V, a partial result on the necessary condition for linearly controllable systems is proved. In Section VI, the local accessibility of nonlinear systems is addressed. Results on both sufficient and necessary conditions for local accessibility are introduced and proved.

II. Equilibrium Set and Controllability

Before the introduction of the theory, we use the following simple example to illustrate some basic concepts and ideas. Consider the following system:

$$\dot{x} = x - u^2.$$

The linearization of the system has a positive uncontrollable eigenvalue at the origin $x = 0$. The system cannot be stabilized at $x = 0$ by any state feedback because $x(t)$ approaches $-\infty$ if the initial condition satisfies $x_0 < 0$. On the other hand, the origin is not the only equilibrium point of the system. In fact, given any initial state $x_0 > 0$, we define the control input as

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