Control Bifurcations

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***Abstract*—A parametrized nonlinear differential equation can have multiple equilibria as the parameter is varied. A local bifurcation of a parametrized differential equation occurs at an equilibrium where there is a change in the topological character of the nearby solution curves. This typically happens because some eigenvalues of the parametrized linear approximating differential equation cross the imaginary axis and there is a change in stability of the equilibrium. The topological nature of the solutions is unchanged by smooth changes of state coordinates so these may be used to bring the differential equation into Poincaré normal form. From this normal form, the type of the bifurcation can be determined. For differential equations depending on a single parameter, the typical ways that the system can bifurcate are fully understood, e.g., the fold (or saddle node), the transcritical and the Hopf bifurcation. A nonlinear control system has multiple equilibria typically parametrized by the set value of the control. A control bifurcation of a nonlinear system typically occurs when its linear approximation loses stabilizability. The ways in which this can happen are understood through the appropriate normal forms. We present the quadratic and cubic normal forms of a scalar input nonlinear control system around an equilibrium point. These are the normal forms under quadratic and cubic change of state coordinates and invertible state feedback. The system need not be linearly controllable. We study some important control bifurcations, the analogues of the classical fold, transcritical and Hopf bifurcations.**

***Index Terms*—Control bifurcation, fold control bifurcation, Hopf control bifurcation, normal form, transcritical control bifurcation.**

I. INTRODUCTION

THE theory of normal forms and bifurcations of a parametrized dynamical system is well known [15]. One considers a smooth vector field

$$\dot{x} = f(x, \mu) \tag{1.1}$$

depending on a parameter μ. The equilibria of the vector field are those x_e, μ_e such that $f(x_e, \mu_e) = 0$. Perhaps the most important property of an equilibrium is its stability. In the first approximation, this is determined by the stability of its linear approximating system around x_e, μ_e

$$\dot{\delta x} = \frac{\partial f}{\partial x}(x_e, \mu_e)\, \delta x. \tag{1.2}$$

If all the eigenvalues of $(\partial f/\partial x)(x_e, \mu_e)$ lie in the open left half plane then the system (1.1) is locally asymptotically stable around the x_e, μ_e. If one or more eigenvalues lie in the open right-half plane then the system (1.1) is unstable. If all the eigenvalues lie in the closed left-half plane but some are on the imaginary axis then the first approximation is not decisive, (1.1) may be locally asymptotically stable or unstable, depending on higher degree terms.

The topological character of the equilibria can change at a critical value of the parameter, perhaps two branches of equilibria cross or a branch loses or gains stability. Such a state and parameter is called a bifurcation point of the parametrized vector field. A local *bifurcation* takes place at a parameter value where the system loses structural stability with respect to parameter variations, i.e., the phase portrait around the equilibrium at the critical parameter value is not locally topologically conjugate to the phase portraits around the equilibria at nearby parameter values. If the local linearizations at two equilibria have no poles on the imaginary axis, the same number of strictly stable and the same number of strictly unstable poles then the local phase portraits are topologically conjugate. Therefore a bifurcation is characterized mathematically by one or more eigenvalues of the linearized system crossing the imaginary axis. We restrict our discussion to local bifurcations which we refer to as bifurcations.

A standard approach to analyzing the behavior of the parametrized ordinary differential equations (ODE) (1.1) around a bifurcation point is to treat the parameter as an additional state variable with dynamics $\dot{\mu} = 0$ and to compute the center manifold of the extended dynamics through the bifurcation point and the dynamics restricted to this manifold [15]. The center manifold is an invariant manifold of the differential equation which is tangent at the bifurcation point to the eigenspace of the neutrally stable eigenvalues. In practice, one does not compute the center manifold and its dynamics exactly, in most cases of interest, an approximation of degree two or three suffices. If the other eigenvalues are in the open left-half plane, then this part of the dynamics is locally asymptotically stable and therefore can be neglected in a local stability analysis around the bifurcation point. The bifurcation point will be locally asymptotically stable for the complete dynamics iff the dynamics on the center manifold is locally asymptotically stable. Of course, at some nearby equilibria the dynamics may be unstable.

The next step is to compute the Poincaré normal form of the center manifold dynamics. From its normal form the bifurcation is recognized and understood. Familiar examples are the fold (or saddle node), the transcritical and the Hopf bifurcations. The first two of these depend on the normal form of degree two and the last one depends on the normal form of degree three. The fold and Hopf bifurcations are the only ones that are generic and of codimension 1, i.e., robust with respect to perturbations and depend on a single parameter, so these are the most important.

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